Random embeddings of bounded degree trees with optimal spread

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Joint work with Alp Müyesser, Paul Bastide

Hiking workshop

Dirac-type thresholds

When does $H \subset G$?

- NP-complete (ex: Hamiltonian cycle)

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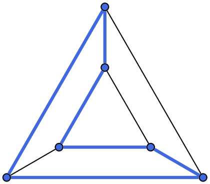
- What about sufficient conditions ?



Dirac 1952

If $\delta(G) \geq n/2$ then G is Hamiltonian

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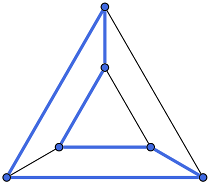

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What is the infimum $\delta_{H,n}$ such that $\delta(G) \geq \delta_{H,n} \Rightarrow H \subset G$?

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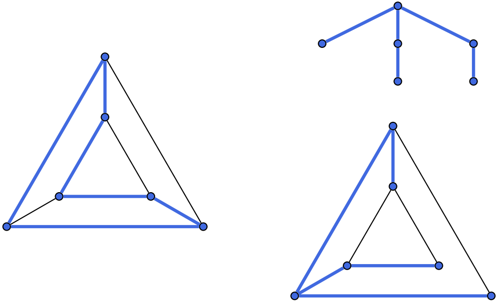

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Focus on Hamiltonian cycles and spanning trees of bounded degree

Counting the embeddings

Embedding

Injection $\phi : H \to G$ such that $uv \in E(H) \Rightarrow \phi(u)\phi(v) \in E(G)$

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If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?

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Sárközy, Selkow, Szemerédi 2003 and Cuckler, Kahn 2003

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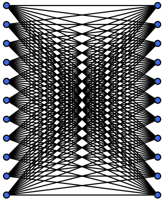

Counting the embeddings

If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?
For which G is this number minimal ?

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Random graphs

Erdős-Rényi graph $G(n, p)$

Keep each edge of K_n independently with probability p

Embedding in a random graph

For what p_n does $\mathbb{P}[H \subset G(n, p_n)] \geq \frac{1}{2}$?

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For what p_n does $\mathbb{P}[H \subset G(n, p_n)] \geq \frac{1}{2}$?

Pósa 1962

$\mathbb{P}[G(n, \log(n)/n) \text{ is Hamiltonian}] \xrightarrow[n\to\infty]{} 1$

Koršunov 1977

Sharp threshold for hamiltonicity

Robustness

Random sparsification $G * p$

Keep each edge of G with probability p

Embedding in a typical subgraph

For what p_n does $\mathbb{P}[H \subset G * p_n] \geq \frac{1}{2}$ for all G with $\delta(G) \geq \delta_{H,n}$?

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Krivelevich, Sudakov, 2014

If $\delta(G) \geq n/2$, then $G * (\log(n)/n)$ remains Hamiltonian with good probability

To sum up

Counting the embeddings

If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?

Embedding in a random graph

For what p_n does $\mathbb{P}[H \subset G(n, p_n)] \geq \frac{1}{2}$?

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Unified approach

q-spread embedding

A distribution $\mathbb{P}$ over embeddings $\phi : H \to G$ is q-spread if $\forall x_1, \ldots x_s \in V(H)$, $\forall y_1, \ldots y_s \in V(G)$,

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Other point of view

Randomized algorithm embedding H progressively, with linearly many options at each step

$$\mathbb{P}[\forall i \leq s, \phi(x_i) = y_i] = \mathbb{P}[\phi(x_1) = y_1] \cdots \mathbb{P}[\phi(x_s) = y_s \mid \phi(x_1) = y_1, \dots \phi(x_{s-1}) = y_{s-1}]$$
$$\leq \left(\frac{C}{n}\right)^s$$

Toy example

Random embedding of an Hamiltonian cycle in K_n

Same as random permutation

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$\forall x_1, \dots x_s \in V(C_n), \forall y_1, \dots y_s \in V(K_n),$

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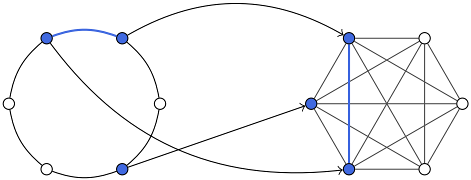

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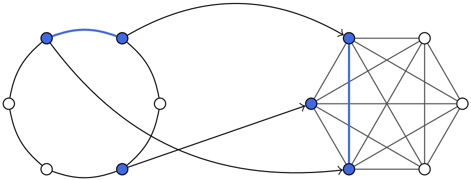
$\forall x_1, \dots x_s \in V(C_n), \forall y_1, \dots y_s \in V(K_n),$

$$\mathbb{P}[\forall i, \phi(x_i) = y_i] = \frac{(n-s)!}{n!} \leq \left(\frac{e}{n}\right)^s$$

by Stirling's formula



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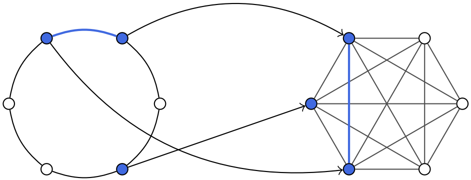

- Same proof for embeddings of spanning trees
- Much harder when G is not a clique

Spreadness implies counting

If there is a q-spread distribution, then for any embedding ϕ_H,

$$\mathbb{P}[\phi = \phi_H] \leq q^{|H|}$$

Hence, $\#$ embeddings $\geq q^{-|H|}$

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Corollary

If there is a $\left(\frac{C}{n}\right)$-spread distribution, then G contains at least $\left(\frac{n}{C}\right)^n$ embeddings of H

Application

$\left(\frac{n}{e}\right)^n$ embeddings of C_n in K_n

Spreadness implies robustness

Kahn-Kalai conjecture 2006

- threshold: $\mathbb{P}(G(n,p) \text{ is Hamiltonian}) \xrightarrow[n\to\infty]{} \begin{cases} 1 & \text{if } p \gg \log(n)/n \\ 0 & \text{if } p \ll \log(n)/n \end{cases}$
- expectation threshold: $\mathbb{E}(\#\text{ Hamiltonian cycles in } G(n,p_E)) \xrightarrow[n\to\infty]{} \begin{cases} > 1 & \text{if } p_E \gg 1/n \\ < 1 & \text{if } p_E \ll 1/n \end{cases}$

For increasing properties, $p_E \leq p = O(p_E \log |H|)$

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Park, Pham 2022

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Kelly, Müyesser, Pokrovskiy 2023

If q-spread distribution of H in G, then $G * (q \log|H|)$ still contains H with good probability

Komlós, Sárközy, Szemerédi 1996

$\forall\Delta, \forall\alpha > 0$, for n large enough, $\delta(G) \geq (\frac{1}{2} + \alpha)n \Rightarrow G$ is universal for spanning trees of maximum degree Δ

Embedding spanning trees of bounded degree

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For all Δ, $G(n, O_\Delta(\log(n)/n))$ is universal for spanning trees of maximum degree Δ

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Pham, Sah, Sawhney, Simkin 23

$O(\frac{1}{n})$-spread distribution for perfect matchings, K_r-factor and spanning trees of bounded degree

Spread distribution on trees of bounded degree

Pham, Sah, Sawhney, Simkin 23

$O(\frac{1}{n})$-spread distribution for perfect matchings, K_r-factor and spanning trees of bounded degree

Bastide, L.-D., Müyesser 25

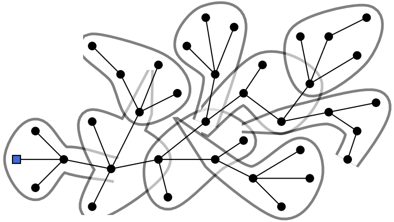
$O(\frac{1}{n})$-spread distribution for spanning trees of bounded degree

- Avoids the Regularity Lemma
- Shorter and more flexible proof
- Better constants
- Generalizes painlessly to hypergraphs and digraphs

Sketch of proof in an ideal world

Chopping T and G

- Split T in subtrees T_i of size $C = O(1)$



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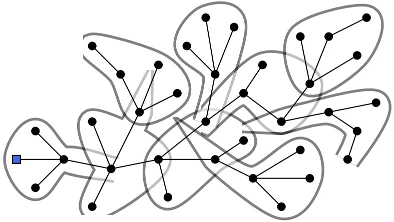

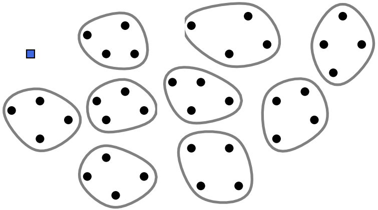

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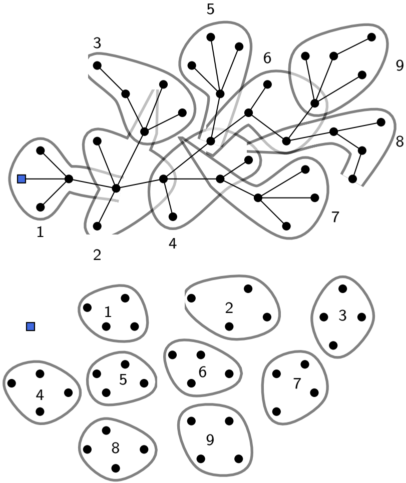

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Assign each T_i a uniform random bag $G_{\phi(i)}$

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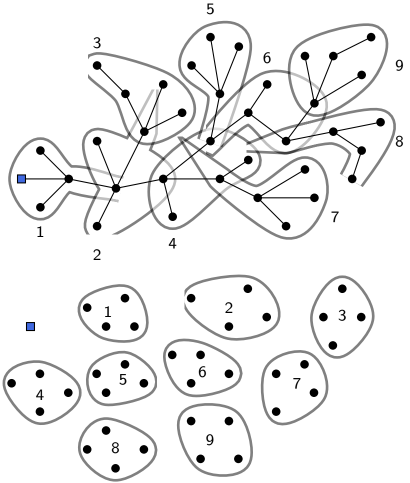

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Embed each T_i deterministically in $G_{\phi(i)}$ using KSS

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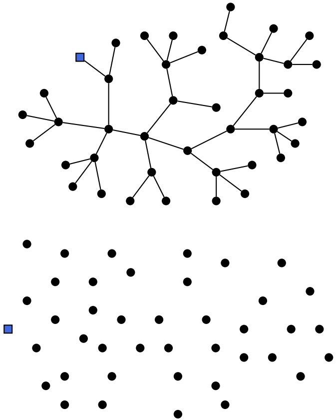
T cannot be split into same size blocks



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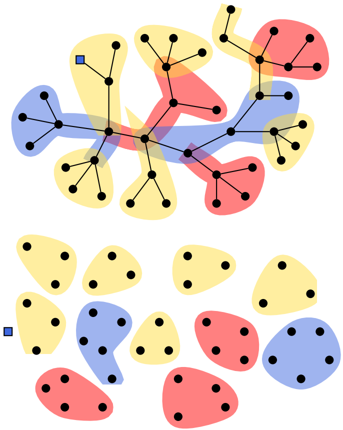

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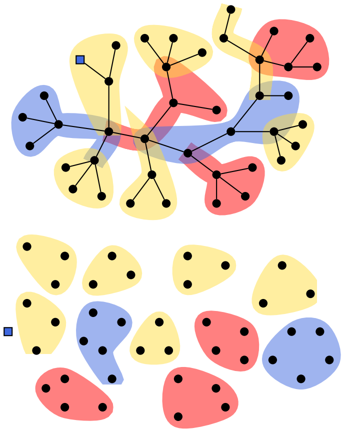

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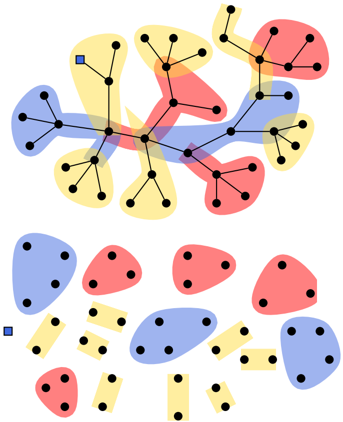
💡 More blocks, slightly smaller, dispatch leftover randomly



$\delta(G') = (1-\varepsilon)|G'|$

Future work

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  Subdivision arguments do not work as nicely
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