Reconfiguration of square-tiled surfaces

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LaBRI, Bordeaux

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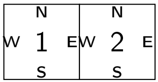
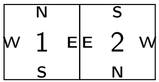
Joint work with Vincent Delecroix.

Square-tiled surfaces

Definition

- Square-tiled surface: gluing of N square tiles on their parrallel sides $\rightsquigarrow$ closed orientable connected surface





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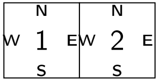
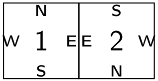
- Quadratic: adjacencies = {NS,EW, NN, SS, EE, WW}
- Abelian: only {NS,EW}



Abelian



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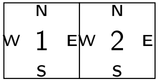

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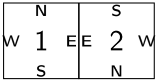

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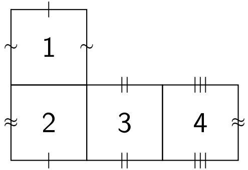
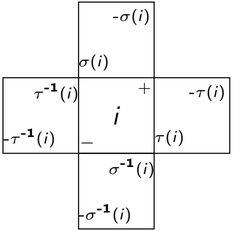


Quadratic encoding

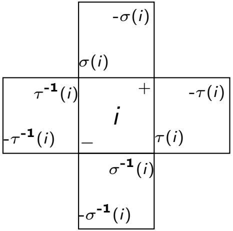


$$\rho = (-1 \;\; +1)\dots(-n \;\; +n)$$

Quadratic encoding

Encoding with involutions

Triplet of involutions without fix-point $\rho, \sigma, \tau \in \mathfrak{S}_{2n}$ that generate a transitive subgroup of $\mathfrak{S}_{2n}$



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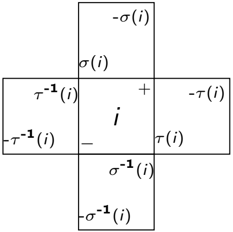

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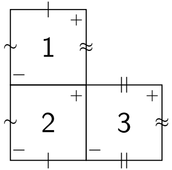
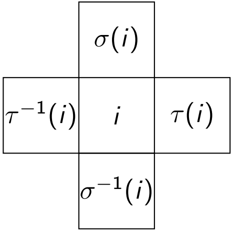


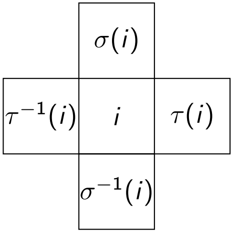
$$\sigma = (-1 \ \ +2)(-2 \ \ +1)(-3 \ \ +3)$$
$$\tau = (-1 \ \ -2)(+1 \ \ +3)(+2 \ \ -3)$$

Abelian encoding



Encoding with permutations

Pair of permutations $\sigma, \tau \in \mathfrak{S}_n$ that generate a transitive subgroup of $\mathfrak{S}_n$



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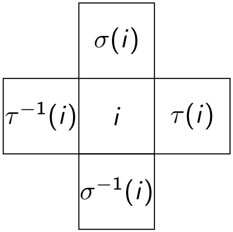
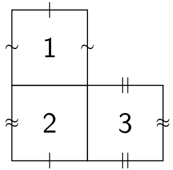
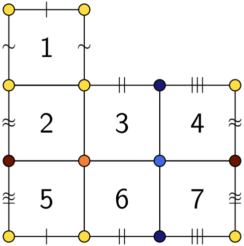




$\sigma = (1\ 2)$
$\tau = (2\ 3)$

Stratum



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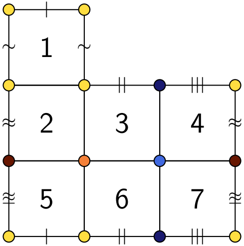
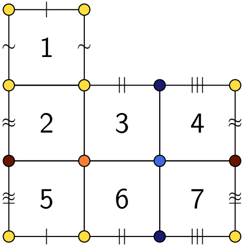


Euler's formula

- μ_i: # vertices of degree $2i$ or angle $i\pi$
- $\sum_i (i-2)\mu_i = 4g - 4$
- Stratum: $[1^{\mu_1}, 2^{\mu_2}, \dots]$

Stratum



$[2^4, 6^1]$ so $g = 2$

Euler's formula

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Monte Carlo methods to sample efficiently

Reconfiguration

- Configuration space $\Omega = ST(\mu)$
- Elementay operation $\leftrightarrow$
- Equivalent configurations: $\exists$ a sequence of operations leading from one to the other
- Reconfiguration graph: Vertices = configurations, edges = elementary operations

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Usual questions

- Are any configurations equivalent ?
- How many reconfiguration steps separate any two configurations ?
- Application to sampling: Does the corresponding Markov chain mix well ?

Markov chains and mixing time

Random Walk P on the reconfiguration graph

- Irreducible: reconfiguration graph connected
- Aperiodic + Irreducible $\Rightarrow$ converges to stationary distribution π
- Symmetric $\Rightarrow$ π uniform

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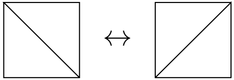
Mixing time

$$t_{mix}(\varepsilon) = \inf\{t : \max_{x \in \Omega} \|P^t(x,\cdot) - \pi\|_{TV} \leq \varepsilon\}$$

where $\|\alpha - \beta\|_{TV} = \sup_{X \subset \Omega} |\alpha(X) - \beta(X)|$

Flips on triangulations

Elementary flip



Disarlo, Parlier 2014

Reconfiguration diameter of n-triangulations of genus g:

- Labeled vertices: $\Theta(g \log(g+1) + n \log(n))$
- Unlabeled vertices: $\Theta(g \log(g+1) + n)$

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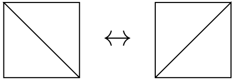

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Budzinski 2018

- For $g = 0$, $t_{mix} = \Omega(n^{5/4})$
- t_{mix} polynomial in n ?

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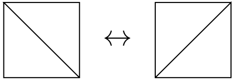

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Not on quadrangulations !

Flips on quadrangulation

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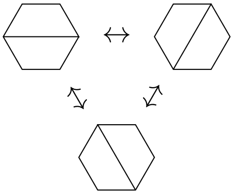

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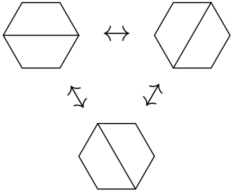

Caraceni, Stauffer 20

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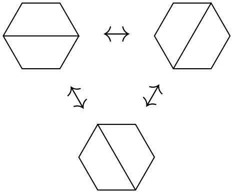

Caraceni, Stauffer 20

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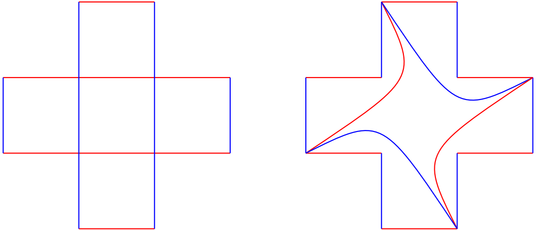
Preserves genus but not square-tiled surfaces !

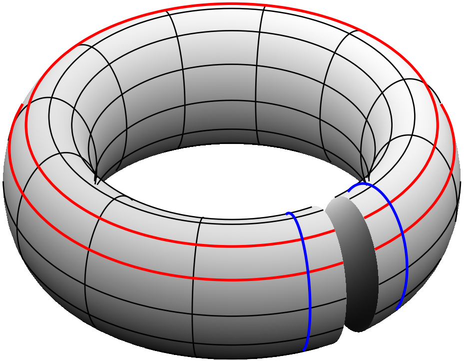
Rotation

Elementary rotation



Preserves genus and square tiled-surface, but not Abelian/quadratic !

Shearing move



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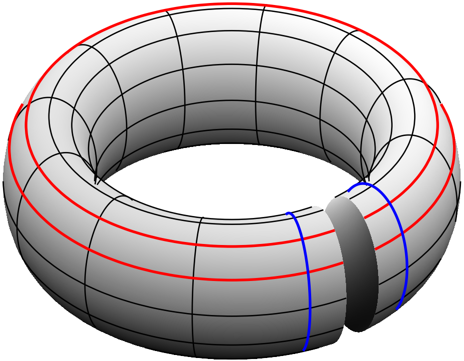

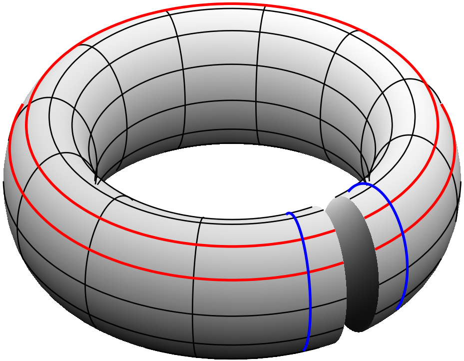

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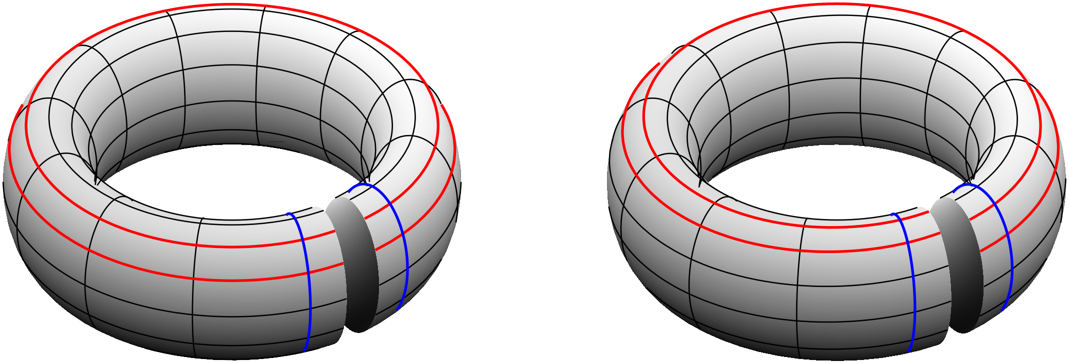

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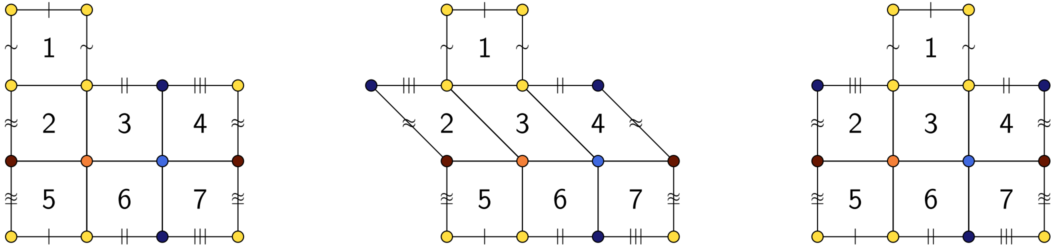

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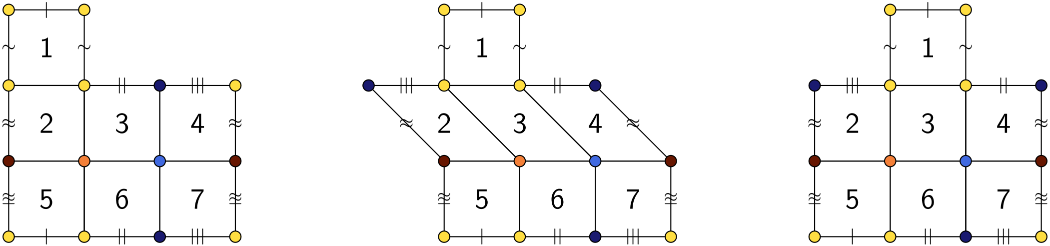

Shearing = multiply σ by a cycle of τ

Shearing moves preserve the angle around the vertices and Abelian property !

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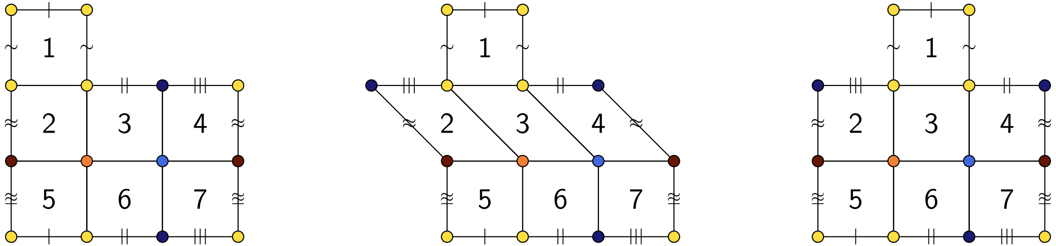

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Two settings

- Slow shears: One shear at a time
- Fast shears: Any number of shears on the same cylinder count as one

Equivalence of square-tiled surfaces

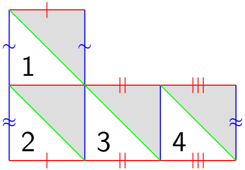
Conjecture [Delecroix, Goujard, Jeffreys, Parlier, Schleimer 2022]

Within any Abelian stratum, all square-tiled surfaces (with identical spin and hyperellipticity) are equivalent

Hyperellipticity

Hyperelliptic square-tiled surface

- Square-tiled surface fixed under rotation of angle π
- Quotient gives a sphere



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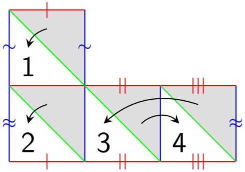

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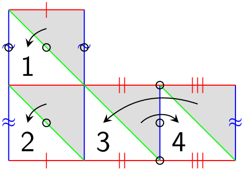

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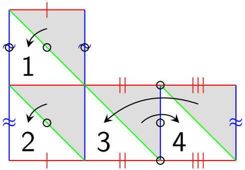

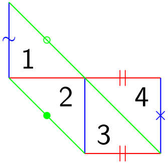

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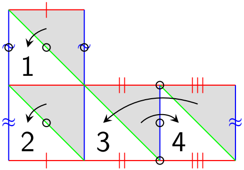

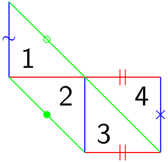

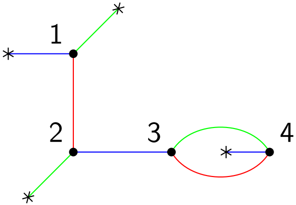

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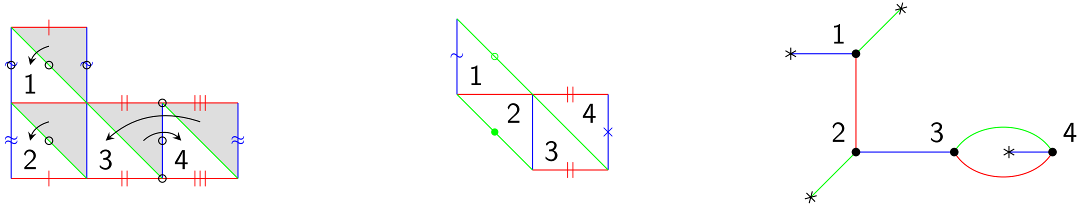
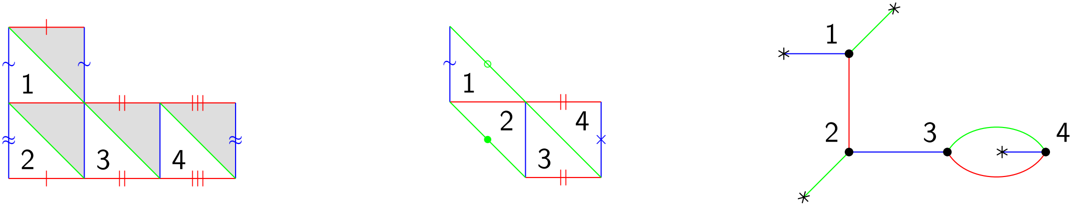


Hyperelleptic component

- $\mu = [2^{\mu_2}, 4g-2]$ or $[2^{\mu_2}, (2g)^2] \rightsquigarrow$ shearing preserves hyperellipticity
- $ST^{hyp}_{Ab}(\mu) \subseteq ST_{Ab}(\mu)$

Strata for tricolored planar graphs



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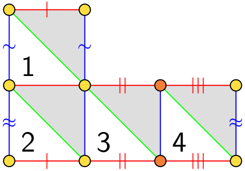
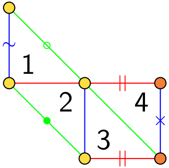
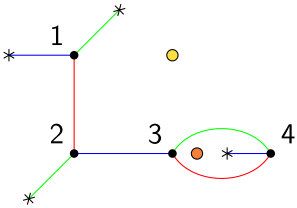


$[2^1, 6^1]$





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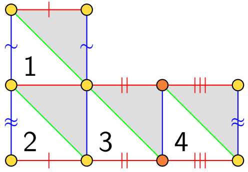

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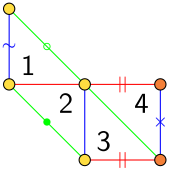
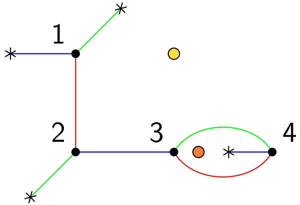


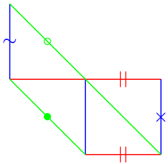


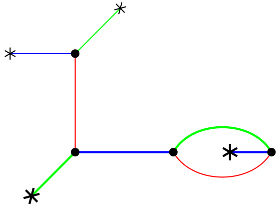
$([1^1, 3^1], 4)$

Stratum

- μ_i: number of faces of degree $3i$
- k: number of triangles
- Euler's formula : $(\sum_i (i-2)\mu_i) - k = 4g - 4 = -4$
- Quotient of $ST_{Ab}^{hyp}(\nu)$: $([1^{\mu_1}, 2^{\mu_2}, d^1], d + 2 - \mu_1)$

Shearing moves in tricolored planar graphs





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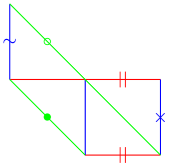

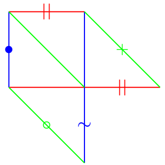

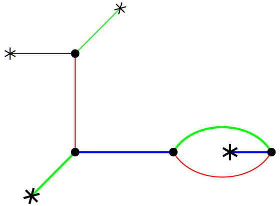

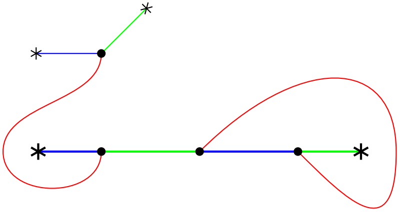

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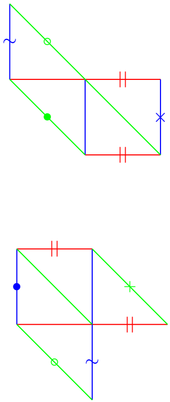
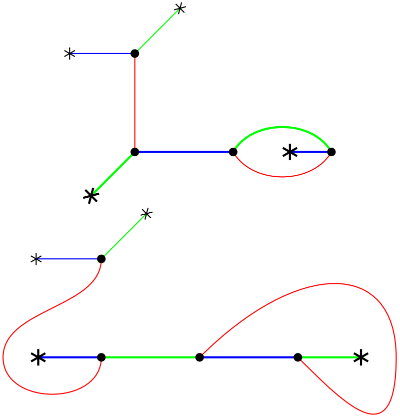




Shearing move

- swap colors + treadmill
- RG and GB in $O(1)$, RB in $O(n)$

Contribution

Delecroix, L. 2023+

Reconfiguration diameter of unlabeled tricolored graphs:

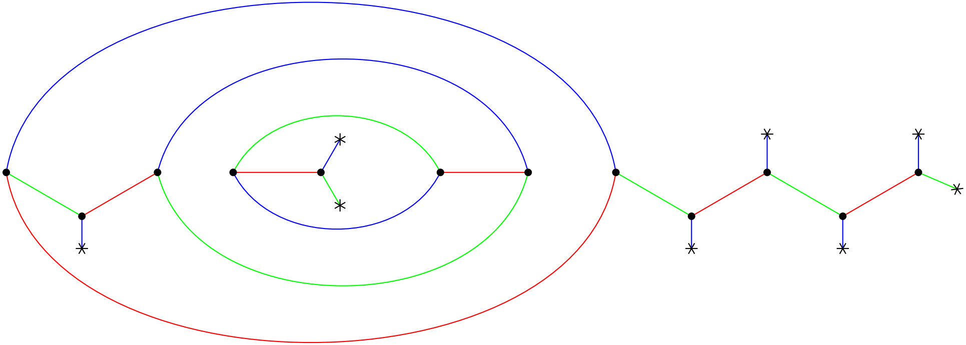
- Abelian hyperelliptic component $ST_{Ab}^{hyp}([2^{\mu_2}, 4g-2])$ and $ST_{Ab}^{hyp}([2^{\mu_2}, (2g)^2])$:

$$O(gn) \text{ slow shears, } \Theta(g) \text{ fast shears}$$

- $g = 0$ and $\mu_1 = 0$:

$$O(kn) \text{ slow shears, } \Theta(k) \text{ fast shears}$$

Reach a “canonical” configuration



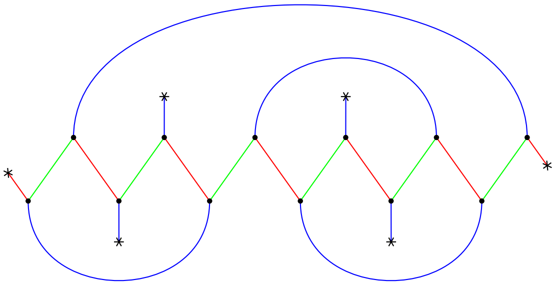
$([2^2, 3^1, 5^1], 8)$

Sketch of proof

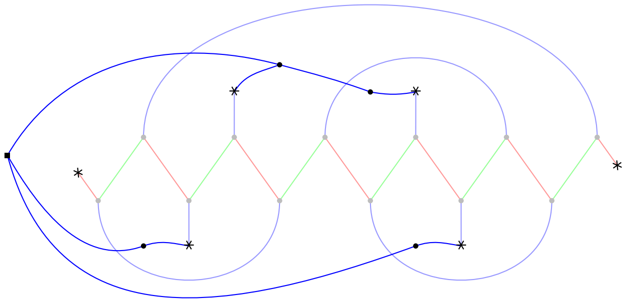
1. Get to a path-like configuration: One RG cylinder finishing with halfedges
2. Reconfiguration within path-likes

1. Take a RG path
2. The RB path at the end of it is a fusion-path
3. Collapse the cylinders with a GB shear.

Blue dual tree



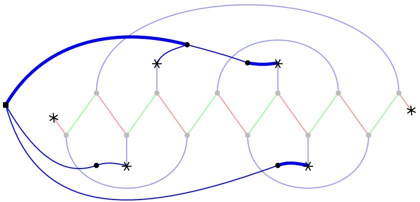
Blue dual tree



Proposition

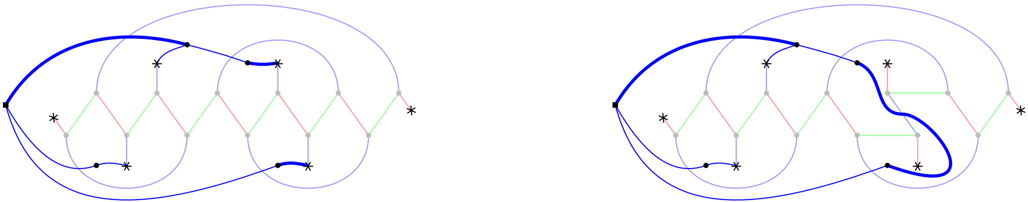
All path-like configurations corresponding to a blue dual tree are equivalent via $O(n)$ RG shears

Reconfiguration of blue dual trees



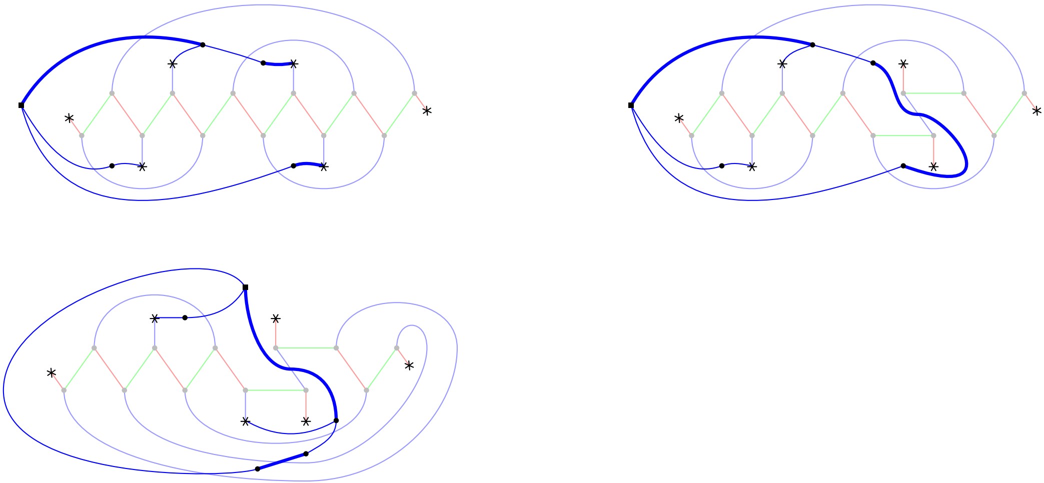
new Glue-cut operation preserving path-likes

Reconfiguration of blue dual trees

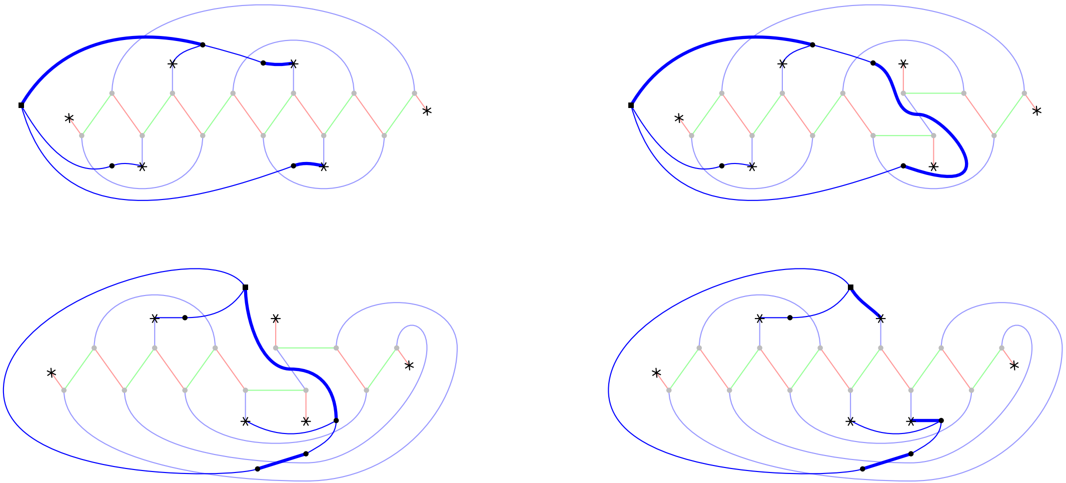


new Glue-cut operation preserving path-likes

Reconfiguration of blue dual trees



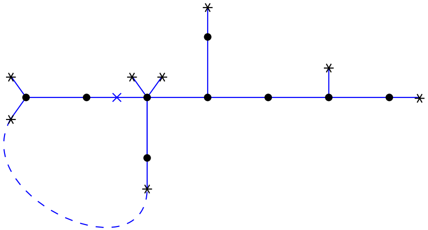
new Glue-cut operation preserving path-likes



new Glue-cut operation preserving path-likes

Using the Glue-cut operation to reconfigure

1. Blue dual tree → Blue dual path
2. Sort the vertices on the path



Future work

Rapid mixing in ST_{Ab}^{hyp} ?

- Among path-like configurations with the glue-cut operation ?
- In general ?

Connectivity in the general case

- Non planar $\Rightarrow$ no dual tricolored planar graph
- Hyperelleptic case negligible, not in all strata

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Thanks !