First Moment method

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Exercise 1.

Let G be a graph with a matching on m edges. Prove that G has a bipartite subgraph H with at least $\frac{1}{2}(|E(G)| + m)$ edges.

Solution 1.

Let (A, B) be the random bipartition constructed as follows. For each edge uv of the matching, toss a fair coin and place u in A with probability $1/2$. Place v in the other side of the bipartition. Dispatch the remaining vertices uniformly at random within A and B. Let H be the bipartite graph induces by the bipartition (A, B). Every edge of the matching belongs to H with probability 1. Every other edge belongs to H with probability $1/2$. Thus, the expected number of edges in H is $\frac{1}{2}(|E(G)| - m) + m = \frac{1}{2}(|E(G) + -m)$.

Exercise 2. Graphs with large chromatic number and girth

Prove that for every integers k and g, there exists a graph with chromatic number at least k and girth at least g.

Exercise 3. List colouring bipartite graphs

Let G be an n-vertex bipartite graph. Let $L : V(G) \to 2^{\mathbb{N}}$ be a map assigning a list of more than $\log_2(n)$ colours to each vertex. Prove that there exists a proper colouring of G such that each vertex uses a colour from its list.

Solution 3.

Let (A, B) be a random partition of the colours in $\bigcup_v L(v)$. We will colour the vertices on one side of the bipartition of G with A and the other side with B. So it suffices to show that with good probability, each vertex is *happy*, i.e. has a colour in A (or in B depending on the side of the bipartition of G). For every vertex v, $\mathbb{P}[L(v) \cap A = \emptyset] < 1/2^{\log_2 n} \leq 1/n$. So the expectation of the number of unhappy vertices is

$$\begin{aligned} \mathbb{E}[\#\text{ unhappy vertices}] &\leq n\mathbb{P}[L(v) \cap A = \emptyset] && \text{by Linearity of expectation} \\ &< n \cdot 1/n \leq 1 \end{aligned}$$

So the probability that all vertices are good is positive by first moment principle.

Exercise 4.

Prove that for n large enough, there exists a graph with chromatic number at least $n/2$ and clique number at most $n^{3/4}$.

Solution 4.

If H is triangle-free, then the complement of H has independance number at most 2, to chromatic number at least $n/2$. So we are looking for a triangle-free graph with independance number at most $n^{3/4}$.

Let $H = G(n, p)$ with $p = n^{-a}$ with $2/3 < a < 3/4$. Let$T$ denote the number of triangles in H, we have

$$\begin{aligned} \mathbb{E}[X] &\leq \binom{n}{3} p^3 && \text{by Linearity of expectation} \\ &\leq \frac{n^{3-3a}}{3!} = o(n) && \text{because } a > 2/3 \end{aligned}$$

By deleting one arbitrary vertex in each triangle of H, we obtain a triangle-free graph H' with $m = (1+o(1))n$ vertices.

The probability that a set $S \subset V(H)$ forms an independent set is $(1-p)^{\binom{|S|}{2}} \leq e^{p\binom{|S|}{2}}$. So letting X denote the random variable counting the number of independent set of size $k = m^{3/4}$ in H,

$$\begin{aligned}
\mathbb{E}[X] &\leq \binom{n}{k}(1-p)^{\binom{k}{2}} && \text{by Linearity of expectation} \\
&\leq \binom{n}{k} \exp\left(-p\binom{k}{2}\right) \\
&\sim \left(\frac{ne}{k}\right)^k \cdot (2\pi k)^{-1/2} \cdot \exp\left(-\frac{k^2}{2n}(1+o(1))\right) \cdot \exp\left(-p\binom{k}{2}\right) && \text{By Stirling} \\
&\xrightarrow[n\to\infty]{} 0 && \text{as } a < 3/4
\end{aligned}$$

So by First moment principle, H (and thus H') has no independent set of size $m^{3/4}$ with positive probability.

Exercise 5.

Prove that every n-vertex 3-uniform hypergraph with $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

Solution 5.

Let S be a random subset of vertices of G such that each vertex belongs to S independently with probability p. Let N and M be the random variables counting the number of vertices of S and the number of edges in S. We have $\mathbb{E}[N] = np$ and $\mathbb{E}[M] = mp^3$ by linearity of expectation. By deleting from S one vertex in each of the edges belonging to S, we obtain an independent set S'. We have $\mathbb{E}[|S'|] \geq \mathbb{E}[N] - \mathbb{E}[M] = np - mp^3$ by linearity of expectation. We now optimise p: taking $p = \sqrt{n/3m}$ we have $\mathbb{E}[|S'|] \geq \frac{2n^{3/2}}{3\sqrt{3m}}$. By the First moment method, there exists an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

Exercise 6.

Let $p > n > m^2$ be integers with p prime. Let $0 < a_1 < a_2 \dots a_m < p$ be integers. Prove that there exists an integer $0 < x < p$ such that the number

$$(xa_i \mod p) \mod n, \quad \text{for } i \in [m]$$

are pairwise distinct.

Solution 6.

Let X_{ij} be the random variable equal to 1 if $(xa_i \mod p) \mod n = (xa_j \mod p) \mod n$, and 0 otherwise. Since p is prime, $(xa_i \mod p)$ takes every value in $[p-1]$ with equal probability. Moreover, note that $(xa_i \mod p) \neq (xa_j \mod p)$ for every $x \in [p-1]$ because p is prime and a_i, a_j distinct. Hence, $\Delta_{ij} = (xa_i \mod p) - (xa_j \mod p)$ takes values in $\{-(p-1), \dots -1, 1, \dots p-1\}$.

For every $y \in \{-(p-1), \dots -1, 1, \dots p-1\}$, for every $i < j$,

$$\begin{aligned}
\Delta_{ij} = y &\Leftrightarrow x(a_j - a_i) + \left(\left\lfloor \frac{xa_j}{p} \right\rfloor - \left\lfloor \frac{xa_i}{p} \right\rfloor\right) p = y \\
&\Leftrightarrow \underbrace{\left(x(a_j - a_i) - \left\lfloor \frac{x(a_j - a_i)}{p} \right\rfloor\right)}_{x(a_j - a_i) \mod p} + \underbrace{\left(\left\lfloor \frac{x(a_j - a_i)}{p} \right\rfloor - \left\lfloor \frac{xa_j}{p} \right\rfloor + \left\lfloor \frac{xa_i}{p} \right\rfloor\right)}_{\in\{0,-1\}} p = y \\
&\Leftrightarrow x(a_j - a_i) \mod p = y \mod p
\end{aligned}$$

Hence, for all $y \in [p-1]$, $\mathbb{P}(\Delta_{ij} = y) + \mathbb{P}(\Delta_{ij} = y - p) = 1/(p-1)$. As there are $\left\lfloor \frac{p-1}{n} \right\rfloor$ mutiples of n among $[p-1]$, by linearity of expectation, $\mathbb{E}(X_{ij}) \leq \left\lfloor \frac{p-1}{n} \right\rfloor \cdot \frac{1}{p-1} \leq 1/n$. Hence Let $X = \sum_{i \neq j} X_{ij}$, we have by linearity of Expectation

$$\mathbb{E}[X] \leq \binom{m}{2} \frac{1}{n} < 1.$$

Thus by first moment method, there exists an x such that the numbers $(xa_i \mod p) \mod n$ are pairwise distinct.

Exercise 7. Method of Deferred Decisions

Let G be the random graph $G(2n, 1/2)$. Show that the following inductive procedure produces a perfect matching M of G with probability at least $1/3$: Choose an arbitrary unmatched vertex u, if all neighbours of u are already matched the procedure fails, otherwise match u with an arbitrary unmatched neighbour and recurse on the remaining unmatched vertices.

Solution 7.

Note that every edge is considered in at most iteration of the procedure, hence we reveal the coin flip determining whether $uv \in E(G)$ only when trying to match u to v. Let A_i be the event that the procedure fails at the i^{th} iteration. Note that A_i depends on the events $(A_j)_{j<i}$ but the probability of A_i doesn't. At the i^{th} iteration, the $2(n+1-i)$ remaining unmatched vertices induce a random graph G_i containing each edge independently with probability $1/2$. Hence, $\mathbb{P}(A_i|\overline{A_{i-1}}) = 2^{-2(n+1-i)+1} = 2^{-2n-2i-1}$. Thus,

$$\begin{aligned}\mathbb{P}\left(\bigvee_i A_i\right) &\leq \sum_{i=1}^{n} 2^{-2i+1} \\ &\leq \frac{2\cdot(4^{n-1}-1)}{3\cdot 4^{n-1}} < 2/3\end{aligned}$$

Exercise 8.

Let H be a graph and let $n > |V(H)|$ be an integer. Suppose that there exists an n-vertex graph L with t edges and containing no copy of H, where $tk > n^2 \ln n$. Prove that there exists a k-colouring of the edges of K_n with no monochromatic copy of H.

Solution 8.

The main idea is to cover K_n by a k random graphs isomorphic to L. Let $\sigma_1 \ldots \sigma_k$ be independent random permutations of $[n]$. Let X_{uv} be the random variable such that $X_{uv} = 1$ if $\sigma_i(u)\sigma_i(v) \notin E(L)$ for each i and 0 otherwise, let $X = \sum_{e \in E(K_n)} X_e$ be the number of edges of K_n that are not covered by one of the copies $\sigma_i^{-1}(L)$ of L. We have $\mathbb{E}[X_e] = \left(1 - t/\binom{n}{2}\right)^k$. By linearity of expectation, we have

$$\begin{aligned}\mathbb{E}[X] &= \binom{n}{2}\left(1 - t/\binom{n}{2}\right)^k \\ &\leq \frac{n^2}{2} e^{-tk/\binom{n}{2}} \\ &\leq \frac{n^2}{2} e^{-2\ln n} && \text{because } tk/\binom{n}{2} > 2\ln n \\ &\leq \frac{1}{2}\end{aligned}$$

By first moment method, there exists $\sigma_1 \ldots \sigma_k$ such that all edges of K_n are covered by one some $\sigma_i^{-1}(L)$. Colour each edge e of K_n with the smallest i such that $\sigma_i(u)\sigma_i(v) \in E(L)$.

★ Exercise 9. Property B in hypergraphs with bounded number of edges

Denote $m(k)$ the minimum number of hyperedges in a k-uniform hypergraph that is not 2-colourable. The goal of this exercise is to prove the following lower bound:

$$\Omega\left(\frac{2^{k-1}k^{1/3}}{\ln(k)^{1/2}}\right) \leq m(k)$$

Let H be a k-uniform hypergraph with $2^{k-1}\ell$ edges, such that there exists $p \in [0, 1/2]$ with $2\ell(1-p)^k < 1/2$ and $2\ell^2 p(1+p)^{k-1} < 1/2$. Show that the following algorithm colours H properly with positive probability:

Step 1 Colour every vertex independently blue or red with probability $1/2$. An edge of H is *dangerous* if it is monochromatic in this first colouring.

Step 2 For every vertex v that belongs to at least one dangerous edge e, change the colour of v with probability p.

1. Let A_e be the event that $e \in E(H)$ monochromatic in the initial and in the final colouring. Prove that $\mathbb{P}[\bigvee_{e\in E(H)} A_e] \leq 2\ell(1-p)^k$.

2. Given two edges $e, f \in E(H)$ such that $e \cap f \neq \emptyset$, let B_{ef} be the event that e is red in the initial colouring and f blue in the final colouring, or e blue in the initial colouring and f red in the final colouring. Prove that $\mathbb{P}[B_{ef}] \leq 2^{2-2k}p(1+p)^{k-1}$. (Hint: partition f into two parts that are initially coloured red and blue)

3. Deduce that the final colouring is proper with positive probability.

4. Conclude.

5. (Optional) How could we improve this bound?

★ Solution 9.

1. We have for each e, $\mathbb{P}[A_e] = 2 \cdot 2^{-k}(p^k + (1-p)^k) \leq 2^{2-k}(1-p)^k$. Let $A = \bigvee_{e\in E(H)} A_e$,

$$\begin{aligned}
\mathbb{P}[A] &\leq |E(H)| \cdot 2^{2-k}(1-p)^k && \text{by Union bound}\\
&\leq 2^{k-1}\ell \cdot 2^{2-k}(1-p)^k\\
&\leq 2\ell(1-p)^k < 1/2
\end{aligned}$$

2. Let R be a subset of $f \setminus e$ (that will be initially coloured red) and denote $r = |R|$. Let C_{efR} be the event that R is initially coloured red and $e \cup (f \setminus R)$ is initially coloured blue, and coloured red in the final colouring (resp. switch blue and red). We have $\mathbb{P}[C_{efR}] \leq 2 \cdot 2^{-(2k-|e\cap f|)}p^{k-r} \leq 2^{2-2k}p^{k-r}$. So,

$$\begin{aligned}
\mathbb{P}[B_{ef}] &\leq \sum_{R\subset f\setminus e} \mathbb{P}[C_{efR}] && \text{by Union bound}\\
&\leq \sum_{r=0}^{k-1} \binom{k-1}{r} 2^{2-2k}p^{k-r}\\
&\leq 2^{2-2k}p \sum_{r=0}^{k-1} \binom{k-1}{r} p^r\\
&\leq 2^{2-2k}p(1+p)^{k-1}
\end{aligned}$$

3. Let D be the event that the final colouring is non-proper. We have $D = \bigvee_{e\in E(H)} A_e \vee \bigvee_{e,f\in E(H)\colon e\cap f\neq\emptyset} B_{ef}$. As there are at most $|E(G)|^2$ pairs of intersecting edges,

$$\begin{aligned}
\mathbb{P}[D] &\leq 2\ell(1-p)^k + 2|E(G)|^2 \cdot 2^{2-2k}p(1+p)^{k-1} && \text{By Union bound}\\
&\leq 1/2 + 2\ell^2 p(1+p)^{k-1} < 1
\end{aligned}$$

So the final colouring is proper with positive probability.

4. We want $\ell = c\frac{k^{1/3}}{\ln(k)^{1/2}}$ for some $c > 1$. By substituting, we have

$$\begin{aligned}
2\ell(1-p)^k &\leq 2\ell e^{-pk}\\
&\leq 2c\frac{k^{1/3}}{\ln(k)^{1/2}}e^{-pk}
\end{aligned}$$

In order to have $2\ell(1-p)^k < 1/2$, it suffices that $p > \frac{\ln\left(k^{1/3}\ln(k)^{-1/2}(4c)^{-1}\right)}{k}$. Fix $p = \frac{\ln(k^{1/3})}{3k}$. We have

$$\begin{aligned}
2\ell^2 p(1+p)^{k-1} &\leq 2\ell^2 pe^{pk}\\
&\leq 2c^2\frac{k^{2/3}}{\ln(k)}\frac{\ln(k^{1/3})}{3k}\ln(k)\\
&\leq 2c^2\ln(k^{1/3})k^{-1/3} \xrightarrow[k\to\infty]{} 0.
\end{aligned}$$

5. To avoid excessive recolouring, one can consider the vertices one after the other, and recolour only those that are in edges that are monochromatic and still haven't been recoloured.

Exercise 10.

1. Prove that for n large enough, every n-vertex directed graph of minimum outdegree $\delta^+ \geq \log_2 n - \frac{\log_2 \log_2 n}{10}$ contains a directed cycle of even length.

2. Using Exercise 9, show that the result still holds when $\delta^+ \geq \log_2 n - \frac{\log_2 \log_2 n}{10}$ for n large enough.

Solution 10.

1. Let G be an n-vertex directed graph with minimum outdegree $\log_2 n - \frac{\log_2 \log_2 n}{10}$. Let (A, B) be a random uniform partition of the vertices of G. Let X be the random variable counting the number of vertices with at most $\sqrt{\log_2 n}$ outneighbours in the other half of the bipartition. By linearity of expectation, we have $\mathbb{E}[X] = \sum_v 2^{-\deg^+(v)} < n2^{\log_2(n)} < 1$. Hence, by the first moment method, there exists a bipartition such that each vertex has at least one neighbour in the other half of the bipartition. This bipartition spans a bipartite graph of positive minimum degree, which contains a cycle of even length.

2. If $\delta^+ \geq \log_2 n - \frac{\log_2 \log_2 n}{10}$ then the number of vertices with no neighbours in the other half of the bipartition is now logarithmic. To get a better bipartition, consider the hypergraph H on $V(G)$ with edge set $\{\{u\} \cup N(u) \colon u \in V(G)\}$. Every edge of H has size at least $k = \delta^+ + 1$, H has at most n edges (as some may repeat). A proper 2-colouring of H corresponds to a bipartition such that every vertex has a neighbour in the other half of the bipartition. We have

$$\begin{aligned}
\frac{2^{k-1}k^{1/3}}{\ln(k)^{1/2}} &\geq \frac{2^{\log_2 n - \frac{\log_2 \log_2 n}{10}}(\log_2 n - \frac{\log_2 \log_2 n}{10})^{1/3}}{\ln(\log_2 n - \frac{\log_2 \log_2 n}{10} + 1)^{1/2}} \\
&\geq \frac{n}{(\log_2 n)^{1/10}} \cdot \frac{(\log_2 n)^{1/3}}{4(\log_2 \log_2 n)^{1/2}} \\
&= \omega(n)
\end{aligned}$$

Hence, Exercise 9 ensures that H has a proper 2-colouring.