Concentration bounds

Markov's inequality X a non-negative random variable,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t} \quad \text{for every } t > 0$$

Chebyshev's inequality X a random variable,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2} \quad \text{for every } t > 0$$

Chernoff bound $X \sim \mathrm{Bin}(n, p)$,

$$\mathbb{P}(|X - np| \geq t) \leq 2e^{-\frac{t^2}{3np}} \quad \text{for every } t > 0$$

Symmetric Lovász Local Lemma $(A_i)_{i\in[n]}$ a collection of events, $d \in \mathbb{N}$ and $p \in [0, 1)$ such that $ep(d+1) \leq 1$ (or $4pd \leq 1$),

(i) for every i, $\mathbb{P}(A_i) \leq p$,

(ii) every A_i is mutually independent from all but at most d other events.

$$\mathbb{P}\left(\bigwedge_i \overline{A_i}\right) > 0$$

Asymmetric Lovász Local Lemma $(A_i)_{i\in[n]}$ a collection of events, D a dependency graph of $(A_i)_{i\in[n]}$, and $(x_i)_{i\in[n]} \in [0, 1)^n$ such that,

(i) for every i, $\mathbb{P}(A_i) \leq x_i \prod_{j\ :\ ij\in E(D)}(1 - x_j)$.

$$\mathbb{P}\left(\bigwedge_i \overline{A_i}\right) \geq \prod_i (1 - x_i)$$

Simple concentration bound $(T_i)_{i\in[n]}$ a collection of independent trials, $X = f(T_1, \dots T_n)$ and $c > 0$ such that

(i) changing the outcome of any single T_i affects X by at most c.

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t\right) \leq 2e^{-\frac{t^2}{2c^2n}} \quad \text{for every } t > 0$$

Talagrand's inequality $(T_i)_{i\in[n]}$ a collection of independent trials, $X = f(T_1, \dots T_n)$ and $c, r > 0$ such that

(i) changing the outcome of any single T_i affects X by at most c,

(ii) for every s, $X \geq s$ can be certified by fixing the outcomes of at most rs trials.

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t\right) \lesssim 2e^{-\frac{\beta t^2}{\mathbb{E}[X]}} \quad \text{for every } \beta < \frac{1}{8c^2r} \text{ and every } t \gg \sqrt{\mathbb{E}[X]}$$

McDiarmid's inequality $(T_i)_{i\in[n]}$ a collection of independent trials, $(\pi_j)_{j1[m]}$ a collection of independent permutations, $X = f(T_1, \dots T_n, \pi_1, \dots \pi_m)$ and $c, r > 0$ such that

(i) changing the outcome of any single T_i affects X by at most c,

(ii) permuting two elements in any single π_j affects X by at most c,

(iii) for every s, $X \geq s$ can be certified by fixing the outcomes of at most rs trials.

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t\right) \lesssim 2e^{-\frac{\beta t^2}{\mathbb{E}[X]}} \quad \text{for every } \beta < \frac{1}{8c^2 r} \text{ and every } t \gg \sqrt{\mathbb{E}[X]}$$

Azuma's inequality $(T_i)_{i\in[n]}$ a collection of trials, $X = f(T_1, \dots T_n)$ and $(c_i)_{i\in[n]} \in \mathbb{R}_{>0}$ such that

(i) for every i, and every $t_1, \dots, t_i$ and t'_i, $\left|\mathbb{E}\left[X | T_1 = t_1, \dots T_i = t_i\right] - \mathbb{E}\left[X | T_1 = t_1, \dots T_i = t'_i\right]\right| \leq c_i$.

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t\right) \leq 2e^{-\frac{t^2}{2\sum_i c_i^2}} \quad \text{for every } t > 0$$