Derandomisation

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Exercise 1.
Let $A_1, \dots A_n$ be a collection of subsets of $[n]$ such that $\sum_i 2^{1-|A_i|} < 1$.

1. Prove that there exists a 2-colouring of $[n]$ such that no set A_i is monochromatic.
2. Give a deterministic algorithm constructing such colouring in time polynomial in $n + m$.
3. Could we have used the method of small probability spaces?

1 Deterministic $1/2$-approximations for the Maxcut problem

Recall from last exercise session that choosing a random uniform bipartition of a graph G results in a cut of expected weight $|E(G)|/2$, which gives a very simple 1/2-approximation for the Maxcut problem. We will call it *the simple probabilistic* $1/2$*-approximation.*

Exercise 2. Conditional probabilities and greedy algorithm

1. Design a greedy deterministic algorithm for the Maxcut problem. Show that this algorithm is a 1/2-approximation.
2. What is its running time?
3. Derandomise the simple probabilistic 1/2-approximation using the method of conditional probabilities.
4. Compare the two algorithms.

Exercise 3. Method of small probability space

1. Derandomise the simple probabilistic algorithm using the method of small probability space.
2. Argue that the obtained algortihm belongs to the class NC.

2 Congestion minimisation

Given a directed graph G and a sequence $(s_1, t_1), \dots (s_k, t_k)$ of pairwise distinct vertices, a sequence of paths $P_1, \dots P_k$, where P_i contains s_i and t_i, has congestion C if every arc of G is contained in at most C paths. Note that $C = 1$ corresponds to the arc-disjoint path problem.

The Congestion minimisation problems asks to find a sequence of paths with minimal congestion.

★ **Exercise 4.**

1. Encode the congestion minimisation problem by an integer linear program, using one variable for each possible path between some s_i and t_i.
2. Relax this ILP into an LP.
3. (Optional) Argue that this LP can be solved in polynomial time, with a polynomial number of non-zero variables.

4. Show that by using fractional rounding, one obtains a $O(\frac{\log n}{\log \log n})$- approximation. To do so, use the following version of the Chernoff bound:
Let X be a sum of independent random variables X_i, such that each $X_i \in [0,1]$ and $\mathbb{E}[X] \leq \mu$.

$$\mathbb{P}(X \geq (1+\alpha)\mu) \leq \exp\left(-\mu((1+\alpha)\ln(1+\alpha) - \alpha)\right) \quad \forall \alpha > 0$$

5. (Optional) What pessimistic estimator could we use to derandomise this algorithm using the method of conditional probabilities?

Hint : Adapt the proof of the Chernoff bound.

Notations: For every i, denote $\mathcal{P}_i$ the set of paths between s_i and t_i. For every path P between s_i and t_i, let X_P^i be the random binary variable indicating whether the path P is chosen by the randomised rounding. For every arc uv, let $Y_{uv} = \sum_i \sum_{P \ni \mathcal{P}_i} X_p^i$ the congestion of the arc uv.