1) Block designs

An (n,k,r) block design is a family $\mathcal{F}$ of k-elements subsets of $[n]$ such that every r-element subset of $[n]$ is included in exactly 1 element of $\mathcal{F}$.

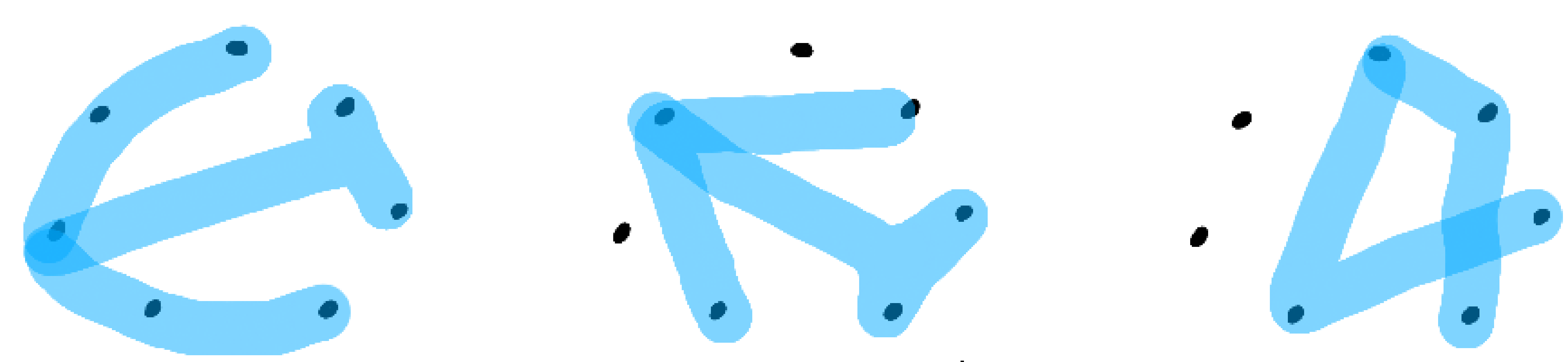
Alternatively, when can the complete r-uniform hypergraph on n vertices be decomposed into copies of the r-uniform hypergraph on k vertices?

- $r=1$: when $k \mid n$.

- $r=2$: graph statement, when can K_n be decomposed into copies of K_k

⇝ $r=2$, $k=3$: when can K_n be decomposed into triangles?

Ex: $n=7$



(Steiner triple system)

1. The degrees are even
2. The number of edges is divisible by 3.

Kirkman 1847 $(n,3,2)$ block designs exists iff $n \equiv 1$ or 3 $[6]$.

Wilson 1978: $(n,k,2)$ block designs exist for large enough n when the obvious divisibility conditions hold.

Conjecture (Steiner 1853): (n,k,r) block designs exist for large enough n, when the obvious divisibility conditions hold.

Keevash (2014-2018) YES (Algebraic techniques + Nibble)

Glock, Kühn, Lo, Osthus (2016) Iterative Absorption

The Rödl - Nibble Method

2) History and Principle

For $2 \le r < k < n$, let

Packing
- $m(n,k,r)$ be the maximal number of edges in a k-uniform hypergraph $\mathcal{H}$ on n vertices s.t. every r-subset of vertices is included in at most one edge:
$$\forall e_1, e_2 \in E(\mathcal{H}),\ |e_1 \cap e_2| < r.$$

Covering
- $M(n,k,r)$ be the minimum number of edges in a k-uniform hypergraph $\mathcal{H}$ on n vertices s.t. every r-subset is included in at least one edge.

We have
$$m(n,k,r) \underset{(1)}{\le} \frac{\binom{n}{r}}{\binom{k}{r}} \underset{(2)}{\le} M(n,k,r).$$

(1) There are $\binom{n}{r}$ r-subsets but each edge has $\binom{k}{r}$ private r subsets.

(2) Every edge covers at most $\binom{k}{r}$ r-subsets and there are $\binom{n}{r}$ such subsets to cover.

Erdős-Hanani 1963 For any fixed r and k,
$$M(n,k,r) \sim \frac{\binom{n}{r}}{\binom{k}{r}} \iff m(n,k,r) \sim \frac{\binom{n}{r}}{\binom{k}{r}}$$

Conjecture
$$m(n,k,r) \sim M(n,k,r) \sim \frac{\binom{n}{r}}{\binom{k}{r}}$$

The idea of the Rödl Nibble method is to perform the random steps sequentially.

Building all the solution in one blow is hard, instead we use a semi-random approach. At each step, build 1% of the solution. By concentration, what remains of the graph verifies roughly the same properties.

Rödl 1985: Proof of the Erdős-Hanani conjecture

Some notation:
- codegree: $\deg(x,y) = \left|\{e \in E(\mathcal{H}) : x,y \in e\}\right|$.
- $\pm\delta$ = any real in $[-\delta, +\delta]$.
- Cover of $\mathcal{H}$ is a set of edges of $\mathcal{H}$ that cover all vertices.

Frankl & Rödl 1985: Generalise Rödl's proof to covers of uniform regular graphs of small codegree.

Pippenger & Spencer 1989: Extend to roughly regular + decomposition into covers i.e. colouring of $E(\mathcal{H})$ with covers.

Pippenger: $\forall r \geq 2$, $\forall k \geq 1$, $\forall a > 0$, $\exists \gamma > 0$, $\exists d_0 > 0$,

$\forall n \geq D \geq d_0$ (large D), Every r-uniform hypergraph $\mathcal{H}$ on n vertices s.t.

roughly D-regular (1) For all but γn vertices, $\deg(x) = (1 \pm \gamma) D$

max degree linear (2) $\forall x \in V$, $0 < \deg(x) < kD$

small codegree (3) $\forall x,y \in V$, $\deg(x,y) < \gamma D$.

contains a cover of at most $(1+a)(n/r)$ edges.

Sketch of proof: Let $\varepsilon > 0$ small.

- A random set of roughly $\frac{\varepsilon n}{r}$ edges w.h.p. covers $O(n\varepsilon^2)$ vertices more than once, so it covers $\varepsilon n - O(n\varepsilon^2)$ vertices.

- After deleting the covered vertices, the remaining hypergraph still verifies conditions (1), (2), (3) for some other n, δ, k, D.

We repeat the procedure. At each step, a roughly ε-fraction of the vertices are covered.

When there are only εn vertices remaining, we take one edge per vertex.

$\blacksquare$

Rödl (1985) $\quad M(n, k, \ell) \leq (1+o(1)) \dfrac{\binom{n}{\ell}}{\binom{k}{\ell}}$.

Proof using Pippenger theorem:

Let $r = \binom{k}{\ell}$ and $\mathcal{H}$ be the hypergraph whose vertices are the ℓ-subsets of n, whose edges are all the collections of $\binom{k}{\ell}$ ℓ-tuples of ℓ-tuples inside a k-set.

- $\mathcal{H}$ has $\binom{n}{\ell}$ vertices
- $\mathcal{H}$ is $\binom{k}{\ell}$-uniform
- $\forall x \in V(\mathcal{H})$, $\deg(x) = D = \binom{n-\ell}{k-\ell}$
- $\forall x, y \in V(\mathcal{H})$, $\deg(x,y) = \binom{n-\ell-1}{k-\ell-1} = o(D)$

So by Pippenger's theorem, $\mathcal{H}$ has a cover of size $(1+o(1)) \dfrac{\binom{n}{\ell}}{\binom{k}{\ell}}$. $\blacksquare$

Lemma (the nibble step):

$\forall r \geq 2, \forall K \geq 1, \forall \varepsilon > 0, \forall \delta' > 0$, $\exists \delta$ and D_0 s.t. $\forall n \geq D \geq D_0$, such that the following hold:

Every r-uniform hypergraph $\mathcal{H} = (V, E)$ on n vertices s.t.

(i) For all vertices $x \in V$ but δn of them, $\deg(x) = (1 \pm \delta) D$
(ii) $\forall x \in V$, $0 < \deg(x) < KD$
(iii) $\forall x, y \in V$, $\deg(x,y) < \delta D$

contains a set E' of edges such that $\mathcal{H}' = (V', E')$ with $V' = V - \bigcup_{e \in E'} e$ satisfies:

(1) $|E'| = (1 \pm \delta) \varepsilon n / r$
(2) $|V'| = n e^{-\varepsilon} (1 \pm \delta')$

(3) For all vertices but at most $\delta' |V'|$ of them
$\deg'(x) = D e^{-\varepsilon(r-1)} (1 \pm \delta')$.

Proof: Let $r \geq 2$, $K \geq 1$, $\varepsilon > 0$ be some fixed constants. $\delta_1, \delta_2 \ldots \geq 0$ tend to 0 as $D_0 \gg \frac{1}{\delta} \to +\infty$.

Construct E' by sampling independently uniformly at random every edge of $\mathcal{H}$ with probability $p = \frac{\varepsilon}{D}$.

- (1): $\to$ $\mathcal{H}$ has $(1-\delta) n$ vertices of degree $\geq (1-\delta) D$
  So $|E| \geq (1-\delta)^2 D n / r$
  $\to \Delta(\mathcal{H}) \leq KD$ so $|E| \leq \frac{(1+\delta) D n + \delta n K D}{r}$

  $\to \mathbb{E}[|E'|] = (1 \pm \delta_1) D n / r$ for some $\delta_1 > 0$
  $\to \mathrm{Var}[|E'|] = |E| \, p(1-p) \leq (1 \pm \delta_1) \frac{\varepsilon n}{r}$

By Chebyshev for some appropriate $\delta_2 > 0$,

$$\mathbb{P}\left(|E'| = (1 \pm \delta_2) \frac{\varepsilon n}{r}\right) > 0.99.$$

(2): Let I_x be the indicator variable that no edge of E' contains x.

$|V'| = \sum_{x \in V} I_x$. Call x good if $\deg(x) = (1 \pm \delta) D$
bad otherwise.

If x is good, $\mathbb{E}[I_x] = \mathbb{P}(I_x = 1)$

$$= (1-p)^{\deg(x)} = \left(1 - \frac{\varepsilon}{D}\right)^{(1\pm\delta)D} = e^{-\varepsilon}(1 \pm \delta_3)$$

If x is bad $0 \leq \mathbb{E}[I_x] \leq 1$. There are at most δn bad vertices so $\mathbb{E}[|V'|] = n e^{-\varepsilon}(1 \pm \delta_4)$.

$$\mathrm{Var}[|V'|] = \sum_{x \in V} \mathrm{Var}[I_x] + \sum_{x \neq y} \mathrm{Cov}[I_x, I_y]$$

$$\mathrm{Cov}[I_x, I_y] = \mathbb{E}[I_x I_y] - \mathbb{E}[I_x]\mathbb{E}(I_y]$$

$$= (1-p)^{\deg(x) + \deg(y) - \deg(x,y)} - (1-p)^{\deg(x) + \deg(y)}$$

$$\leq (1-p)^{-\deg(x,y)} - 1 \leq \left(1 - \frac{\varepsilon}{D}\right)^{-\delta D} - 1 \leq \delta_5.$$

So $\mathrm{Var}\,|V'| \leq \mathbb{E}[V'] + \delta_5 n^2 \leq \delta_6 \; \mathbb{E}[|V'|]^2$

By Chebyshev, with probability 0.99, $|V'| = (1 \pm \delta_7)\, \mathbb{E}[|V'|]$
$= (1 \pm \delta_8)\, n e^{-\varepsilon}.$

(3) Here again Chebyshev. See Exercise session

$\blacksquare$

Proof of Pippenger theorem:

Let $\varepsilon > 0$ s.t. $\dfrac{\varepsilon}{1-e^{-\varepsilon}} + 2\varepsilon < 1 + a$

Fix $0 < \delta < 1/10$ s.t.

$$(1+4\delta)\frac{\varepsilon}{1-e^{-\varepsilon}} + 2\varepsilon < 1+a.$$

Fix t an integer s.t. $e^{-\varepsilon t} < \varepsilon$.

We apply the lemma t times. Let $\delta = \delta_t$ and define by reverse induction $\delta_t > \delta_{t-1} > \dots > \delta_0$ s.t. $\delta_i \leq \delta_{i+1} e^{-\varepsilon(t-1)}$.

and $\prod_{i=0}^{t} (1+\delta_i) < 1 + 2\delta$

3) Colouring graphs of girth ≥ 5 and bounded degree.

Let G be a triangle free graph of max degree Δ:

Last time, we proved $\exists \varepsilon > 0$, $\chi(G) = (1-\varepsilon)\Delta$ for Δ large.

Borodin Kostochka 1977 & Catlin 1978 & Lawrence 1978

$$\chi(G) \leq \frac{3}{4}(\Delta+2)$$

Kostochka
$$\chi(G) \leq \frac{2}{3}(\Delta+2)$$

Johansson 1996: $\chi(G) = O\left(\frac{\Delta}{\ln(\Delta)}\right)$.

Exercise session: $\forall g_0, \forall \Delta_0, \quad \exists G, \quad \text{girth}(G) \geq g_0 \quad \Delta(G) \leq \Delta_0$,

$$\chi(G) \geq \frac{\Delta}{2\ln(\Delta)}(1+o(1)).$$

Kim 1995: If G has girth ≥ 5, $\chi(G) \leq \frac{\Delta}{\ln \Delta}(1+o(1))$

Naive Colouring procedure (U):
- Assign a random available colour to every vertex of U
- Remove the colours at the monochromatic edges.

Idea of Kim's procedure: poly$(\ln \Delta)$ iterations of the Naive colouring procedure on small random sets of vertices.

Why girth 5? L_u and L_v are very dependent if u and v have many common neighbours.
$\rightarrow$ No common neighbour, then $(L_u)_{u \in V}$ are roughly

random independent sets of colours.

→ As L_u and L_v are roughly independent, when $|L_u| \ll C$
L_u and L_v have few intersections so the final colours of
u and v are almost independent (if $uv \notin E$)
⇝ the colours of an independent set resembles a random uniform independent assignment.

→ P(u gets coloured) ≈ Coupon Collector problem.

$$\Delta \approx C \ln C \quad \text{i.e.} \quad C \approx \frac{\Delta}{\ln(\Delta)}.$$

The Nibble procedure:

1. For each uncoloured vertex v, activate v with probability p
2. For each activated vertex v, assign v a random uniform colour from L_v.
3. Remove the colour assigned to each v from L_u for $u \in N(v)$.
4. Uncolour the monochromatic edges.
5. Equalizing coin flip: $\forall v$, $\forall c \in L_v$, remove c from L_v depending on a coin flip'.

Obs1: • Step 1 insures that more vertices retain their colour.

• As most vertices retain their colours, Step 3 does not hurt us.

• Step 5 simplifies computation.