The Rödl-Nibble method

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Exercise 1. Coupon Collector

Suppose you are collection coupons, coloured with colours $\{1, \dots C\}$. Prove that for every $\varepsilon > 0$,

- after collecting $(1+\varepsilon)C \ln C$ coupons, one collected coupons of every colour with probability $1 - o(1)$ (as C tends to infinity)
- after collecting $(1-\varepsilon)C \ln C$ coupons, misses one colour with propability $1 - o(1)$.

1 Independent sets in triangle-free graphs of bounded average degree

The average degree of a graph G is $\operatorname{avgdeg}(G) = \frac{2|E(G)|}{|V(G)|}$.
Ajtai, Komlós and Szemerédi proved in 1981 the following theorem:

Theorem 1. *Every triangle-free graph G on n vertices, with average degree at most d has an independent set of size $\Omega(\frac{n \log d}{d})$.*

Exercise 2.

Give an example of graphs showing that the triangle-free assumption is necessary.

Exercise 3.

Give a rough sketch of how to use the Nibble method to prove Theorem 1.

Exercise 4.

Deduce an upper bound on the off-diagonal Ramsey number $R(3, t)$ from Theorem 1.

Exercise 5. The Nibble step of Theorem 1

Let G be a n-vertex graph with average degree d and maximum degree $\Delta \leq 10d$. Prove that there exists a subset S of vertices such that the following conditions are verified.

- $|S| \geq \frac{n}{100d}$
- The set S spans at most $\frac{|S|}{50}$ edges,
- At least $n/2$ vertices are at distance at least 2 from S.
- Let H be the graph induced by the vertices that are neither in S nor neighbours of S. Let n', e' and d' be the number of vertices, of edges and the average degree of H. We have

$$\frac{n'^2}{2e'} = \frac{n'}{d'} > \nu \frac{n}{d} = \frac{n^2}{2e} \quad \text{where } \nu = 1 - 1/d - O(\sqrt{d/n})$$

Hint: For the last point, use the following inequality. For every n-vertex triangle-free graph of maximum degree Δ, for every $x \in [0, \frac{1}{10\Delta}]$,

$$\frac{1}{|E(G)|} \cdot \sum_{uv \in E(G)} e^{-x(\deg(u)+\deg(v))} \leq \left(\frac{1}{n} \cdot \sum_{u \in V(G)} e^{-x \deg(u)} \right)^2 \tag{1}$$

2 Rödl's theorem on approximate block designs

★ Exercise 6. Pippenger

The goal of this exercise is to prove the Nibble step of Pippenger's theorem. Prove the following:

For every integer $r \geq 2$, reals $K \geq 1$, $\varepsilon > 0$ and $\delta' > 0$, there exists $\delta = \delta(r, K, \varepsilon, \delta')$ and $D_0 = D_0(r, K, \varepsilon, \delta')$ such that the following holds. For every n and D such that $n \geq D \geq D_0$, for every n-vertex r-uniform hypergraph $\mathcal{H}$ such that

(i) For every vertex $x \in V(\mathcal{H})$ but at most δn of them, $\deg(x) = (1 \pm \delta)D$,

(ii) For every vertex $x \in V(\mathcal{H})$, $0 < \deg(x) < KD$

(iii) For every distinct vertices x and $y \in V(\mathcal{H})$, $\deg(x, y) \leq \delta D$

there exists a set E' of edges such that

1. $|E'| = (1 \pm \delta')\frac{\varepsilon n}{r}$
2. The set $V' = V(\mathcal{H}) \setminus \bigcup_{e \in E'} e$ has size $(1 \pm \delta')ne^{-\varepsilon}$
3. In the subhypergraph $\mathcal{H}'$ induced by V', for every vertex x but at most $\delta'|V'|$ of them, $\deg_{\mathcal{H}'}(x) = (1 \pm \delta')De^{-\varepsilon(r-1)}$.