Absorption method

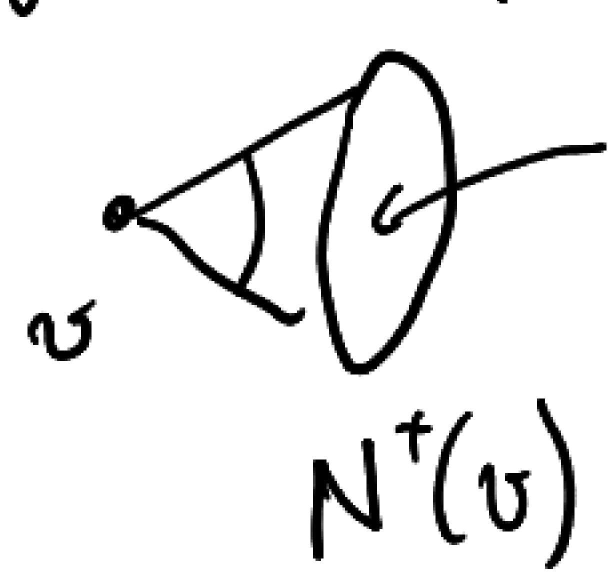
1) A non-probabilistic example

Erdős 1986 For every k, there exists N such that for every $n \geq N$ divisible by k, every n-vertex tournament can be partitioned into transitive subtournaments of size k.

[Erdős - Moser 1963] $\forall n \geq 2^{k-1}$, any n-vertex tournament contain an k-vertex transitive tournament.

Proof: Induction on k.

Take an arbitrary vertex v. We have $N^+(v)$ or $N^-(v) \geq 2^{k-2}$ by pigeon hole principle, say $N^+(v) \geq 2^{k-2}$



Apply induction. ■

Note that we could get $n \geq 4^k$ by Ramsey by ordering the vertices arbitrarily and colouring blue the edges $\rightarrow$
red $\leftarrow$

Proof of Erdős 1986:

Let T be a large tournament.

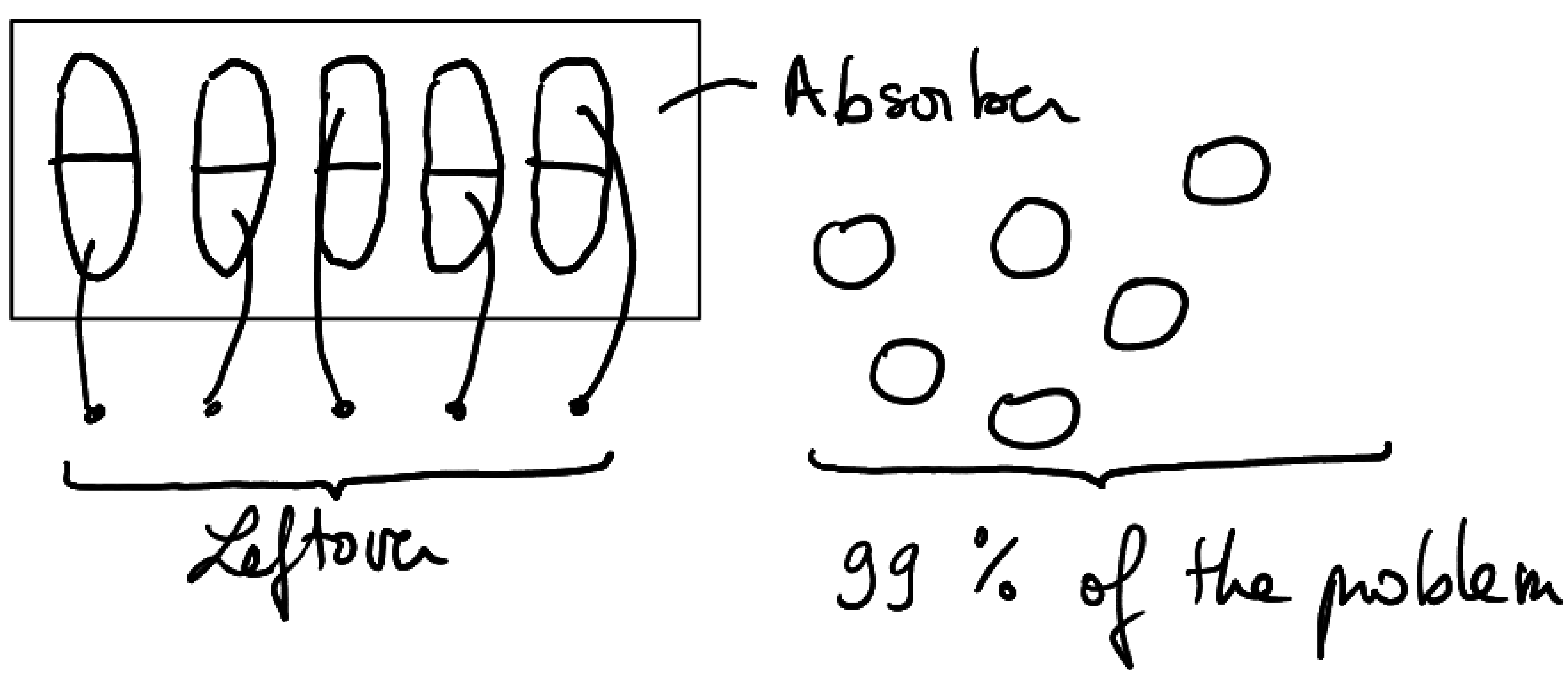
Step 1: Put aside $K = k2^{k-1}$ disjoint transitive tournaments of size $2k-1$.

Step 2: While there are more than K vertices left, pluck

and remove transitive tournament of size k.
As $k \mid n$, this stops when there are exactly K vertices left.

Step 3: Match each remaining vertex v_i with a tournament T_i of size $2k-1$.

Rearrange: Say $|N^+(v_i) \cap T_i| \geq k-1$ then v_i and $k-1$ of its outneighbours form a transitive tournament. $\blacksquare$



Note: this was the first application of the absorption method.

2) Principle

That's the absorption method:

Step 1 (Preparation): Set aside a small absorber, i.e. a flexible structure that satisfies the constraints in a rich way.

Step 2 (99% of the solution) In the remainder, construct 99% of the solution

Step 3 (Absorption): Combine the leftover of step 2 with the absorber. Using small modifications, absorb the leftover.

We typically use probabilistic method in Step 1 and PM or Ramsey in Step 2.

This works when:

(A) It is significantly easier to construct some object satisfying 99% of the constraints

(B) the partial solution are robust enough that they can be locally modified to produce other partial solutions.

Typically for tilings / spanning structures / decompositions.

3) McDiarmid concentration inequality:

McDiarmid 2002: Let $X = f(T_1, \dots T_n, \Pi_1, \dots \Pi_m)$ where the T_i's are independent trials and $\Pi_1, \dots \Pi_m$ independent permutations s.t. $\exists c, r > 0$:

(i) Changing the outcome of one T_i affects X by at most c,

(ii) interchanging two elements in one permutation affects X by at most c,

(iii) $\forall s$, there is a set of at most rs choices whose outcomes certify $X \geq s$.

Then $\forall t \in [0, \mathbb{E}[X]]$,

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq t + 60c\sqrt{r\,\mathbb{E}[X]}\right) \leq 4\, e^{-\frac{t^2}{8c^2 r\,\mathbb{E}[X]}}$$

The case $n = 0$, $m = 1$ (which we will use today) was proven by Talagrand.

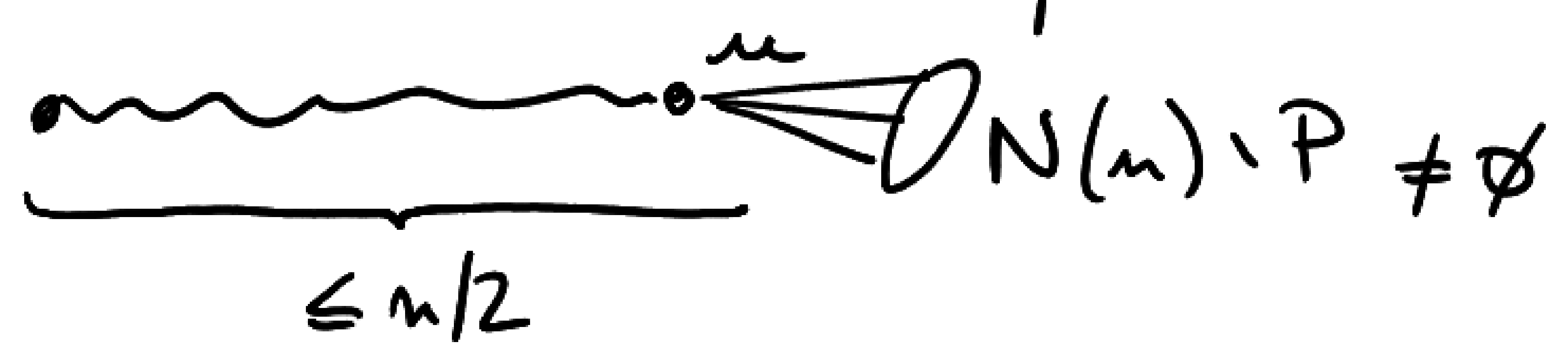
4) Dirac-type problems

Codification of the absorption method by Rödl, Ruciński & Szemerédi in 2006 to generalise classical Dirac theorems to hypergraphs.

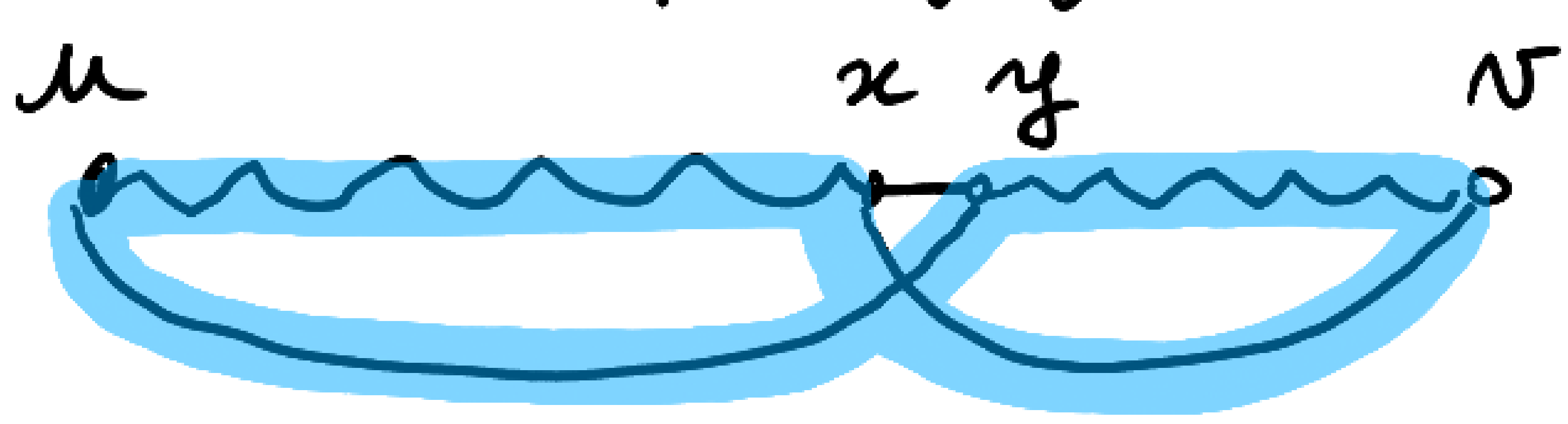
Dirac's theorem 1951 If $\delta(G) \geq \frac{n}{2}$, where $|G| = n$, then G has a Hamiltonian cycle (and a perfect matching).

Proof by switching method:

Step 2 (50% of the solution): G contains a maximal path on at least $\frac{n}{2}+1$ vertices:



Step 3 (handling the leftover): If P is a maximal path then $V(P)$ contains a spanning cycle:



$N(u) \subseteq V(P)$
$N(v) \subseteq V(P)$

By Pigeon hole principle, $\exists x, y$ consecutive on P where $x \in N(v)$, $y \in N(u)$. We do a Posá rotation to obtain a spanning cycle.

If $V(P) \subsetneq V(G)$, create a longer path:

■

Theorem: For every $\varepsilon > 0$, for n sufficiently large, any n-vertex graph with $\delta(G) \geq (1+\varepsilon)\frac{n}{2}$ has an Hamiltonian cycle.

Why? • This can be extended to hypergraphs (ex: Rödl, Ruciński, and Szemerédi 2006) but simpler to prove
• Introduce the notions reservoir

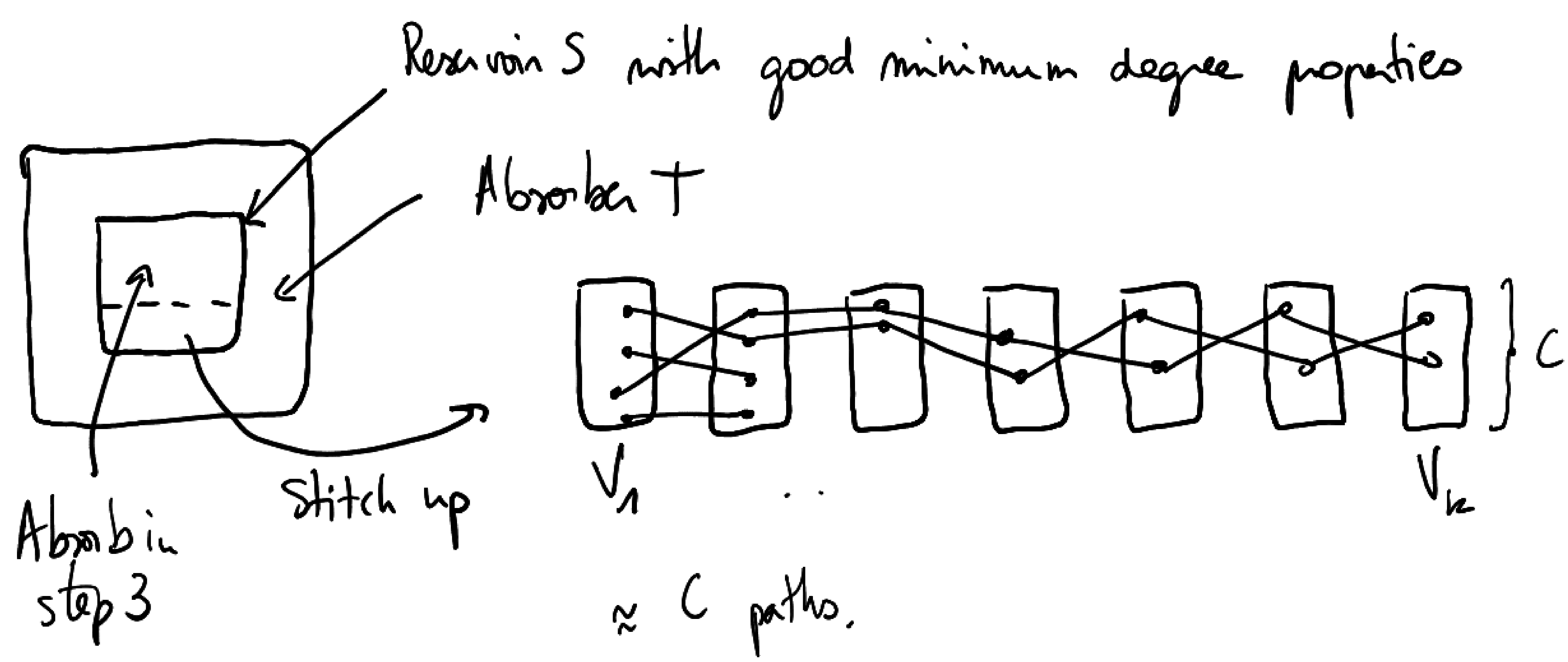
Sketch of proof:

1. Set aside a random subset S of size $n^{1/100}$
   W.h.p. it has good minimum degree inside and with the rest.

   Use this reservoir to build a small absorber T.

2. Take a random partition $V_1, \dots V_k$ into sets of size C of what remains.
   W.h.p: there is a perfect matching between V_i and V_{i+1} for every i ⤳ C paths of length k.

3. Stitch up the paths using the reservoir
   Absorb the leftover of the reservoir.



Lemma 1: Let A, B be disjoint random subsets of size C in G (where $\delta(G) \geq (1+\varepsilon)\frac{n}{2}$).

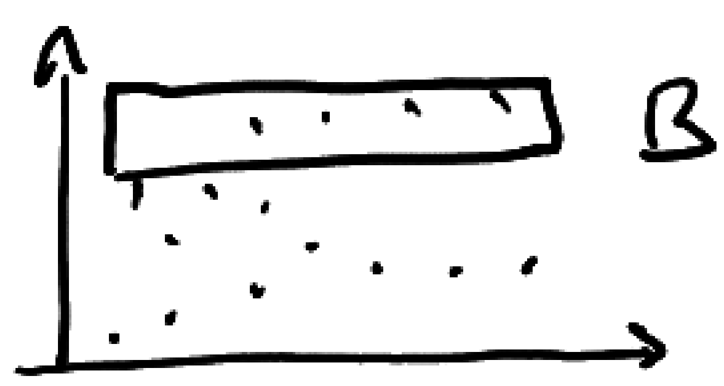
$$\mathbb{P}(\exists \text{ a perfect matching between } A \text{ and } B) \geq 1 - 4C e^{-\varepsilon^2 \Omega(C)}$$

Proof: Let $v \in A$ and $X_v = |N(v) \cap B|$.

$$\mathbb{E}[X_v] \underset{\text{L.E.}}{=} \frac{C}{n}|N(v)| \geq (1+\varepsilon) C.$$

If B was a random subset where each vertex is sampled independently, we could use Chernoff. Here we don't have independence because we specify the size of B.

Let π be a random permutation of $V(G) \setminus u$ and B



be the C vertices with largest $\pi(u)$.

(ii) Permuting two elements of π affects X_v by at most 1 $\rightsquigarrow c = 1$

(iii) $\forall s$ setting $\pi(u)$ for the s neighbors of v certifies $X \geq s$. $\rightsquigarrow r = 1$

$$\mathbb{P}\left(X_v < \left(1+\frac{\varepsilon}{2}\right)C\right) \leq \mathbb{P}\left(|X_v - \mathbb{E}[X_v]| \geq \frac{\varepsilon}{2}C\right)$$

$t \sim \varepsilon C \gg \sqrt{C}$

$$\underset{\text{McDiarmid}}{\leq} 4 e^{-\Omega\left(\frac{\varepsilon^2 C^2}{C}\right)}$$

so $\mathbb{P}\left(\forall v \in A,\ X_v \geq \left(1+\frac{\varepsilon}{2}\right)C\right) \geq 1 - \mathbb{P}\left(\exists v \in A,\ X_v \leq \left(1+\frac{\varepsilon}{2}\right)C\right)$

$$\underset{\text{Union Bound}}{\geq} 1 - 4C e^{-\varepsilon \Omega(C)}. \qquad \blacksquare$$

Step 2: $C \sim (\log n)^2$

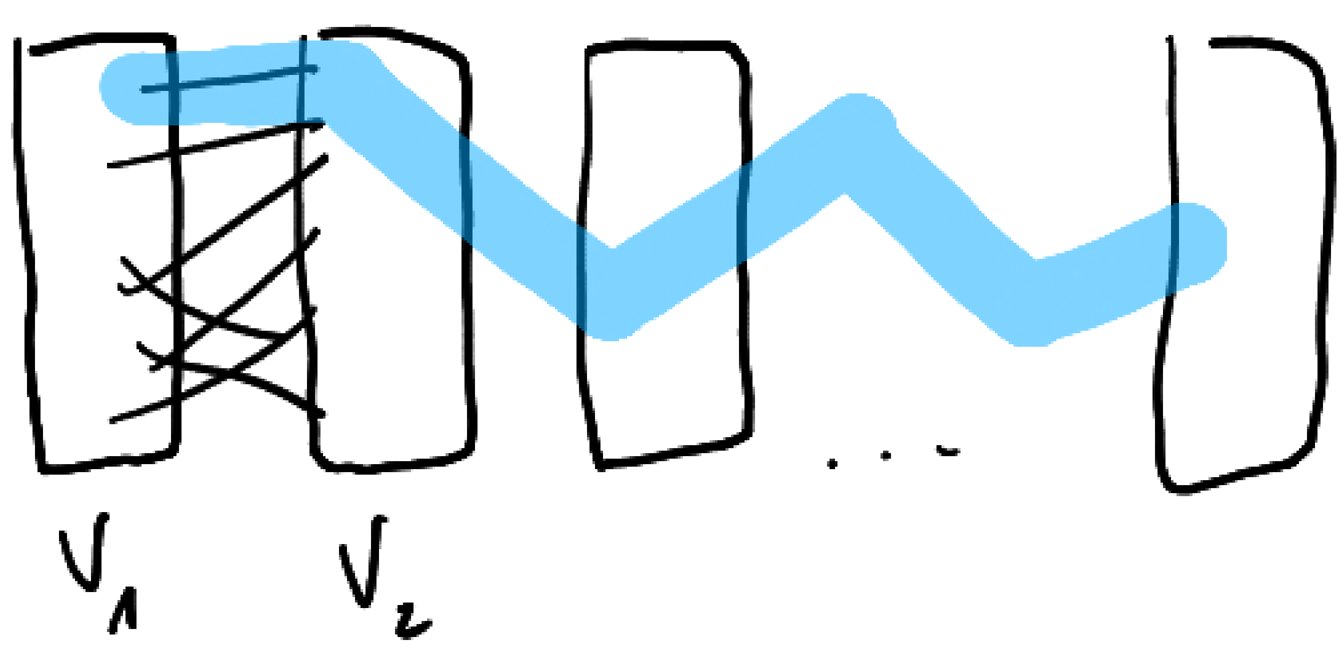
Let V' be a subset of $m = (1-o(1))n$ vertices, s.t. $C \mid m$.

Partition V' randomly into $V_1, \dots V_{\frac{m}{C}}$ sets of size C.

$$\mathbb{P}\left(\forall i,\ (V_i, V_{i+1}) \text{ contains a perfect matching}\right)$$

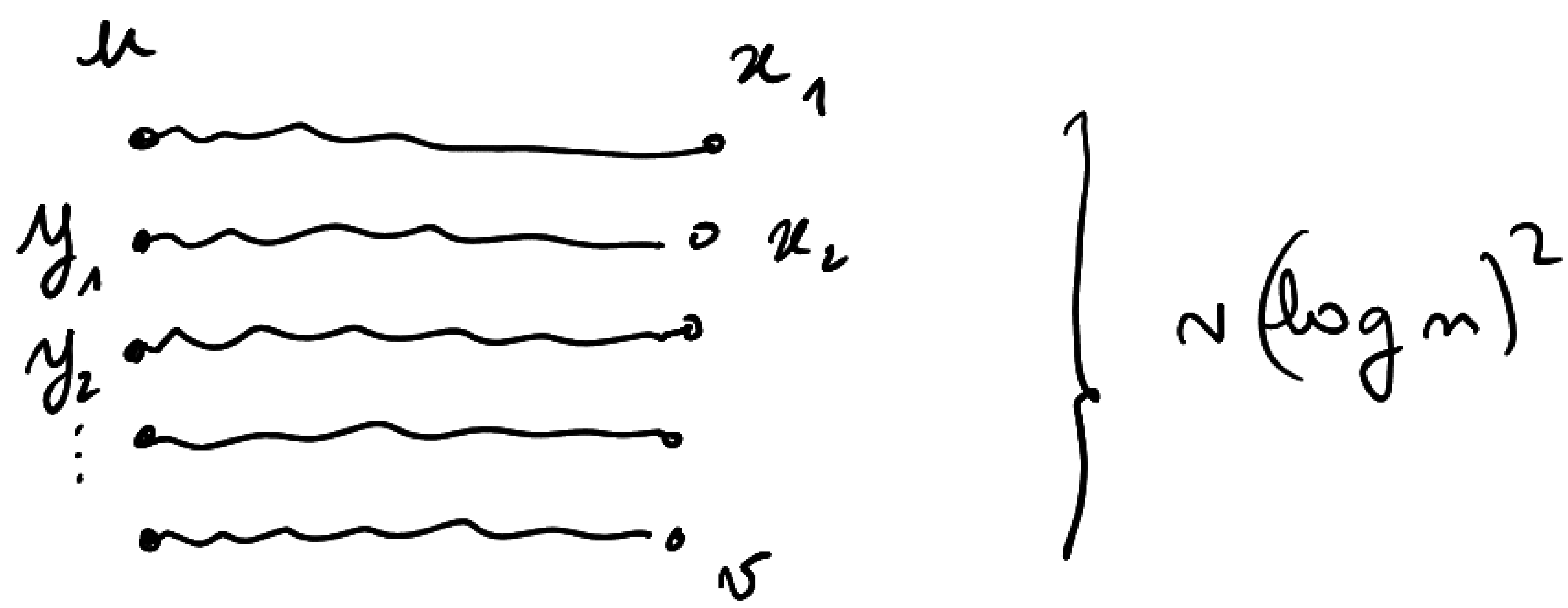
$$\geq 1 - 4\,\frac{m}{C}\, C\, e^{-\varepsilon^2 \Omega(C)}$$

$$\geq 1 - 4\,\frac{n}{n^{\varepsilon^2 \Omega(\log n)}} = 1 - o(1).$$



So with probability $1-o(1)$, we can decompose V' in C vertex disjoint paths.

Fix u and v two vertices. Up to splitting some paths, we can assume that u and v are extremities of distinct paths



Building the absorber:

Step 1a.: The Reservoir.

Lemma 2 Let G with $\delta(G) \geq \frac{(1+\varepsilon)}{2} n$, let $S \subseteq V(G)$ be a random subset of size D sampled uniformly at random. With probability $1 - n^2 e^{-\varepsilon \Omega(D)}$ the following holds:

$$\forall u, v \in V(G),\ |N(u) \cap N(v) \cap S| \geq \frac{\varepsilon D}{4}$$

Proof: Similar to the previous proof, do it in exercise session ■

We take $D \sim n^{1/100}$

Step 1b: The Absorber

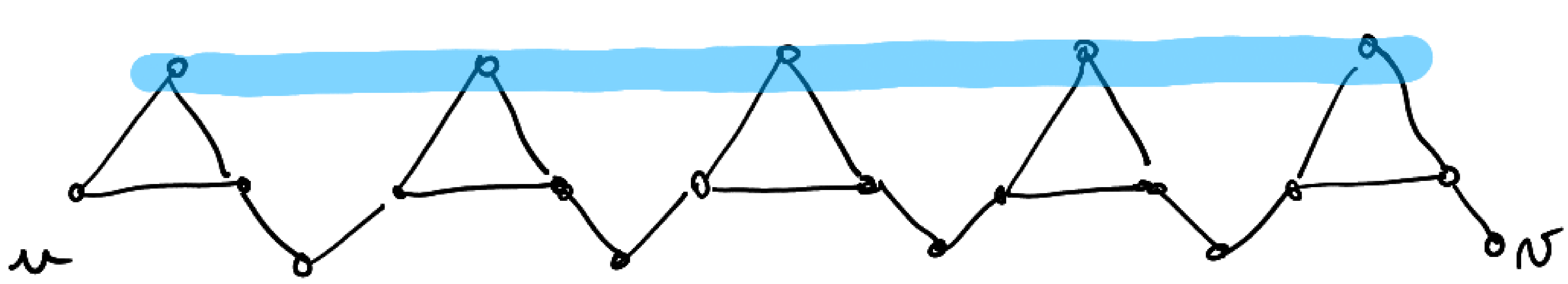
Let T be a subgraph of G, let $u, v \in V(T)$ and $S \subseteq V(T)$.

T is a $(\{u,v\}, S)$ absorber if $\forall S' \subseteq S$, there exists a spanning path of $T \setminus S'$ that has spanning path with extremities u and v.

Lemma 3 $\delta(G) \geq \frac{(1+\varepsilon)}{2} n$ and $|S| \leq \frac{\varepsilon n}{10}$, $u, v \in V(G) \setminus S$.

There exists a $(\{u,v\}, S)$ absorber T, where $T = 4|S|$.

Proof: We first find a collection of disjoint triangles, each containing exactly one vertex of S. Such collection exists by pigeonhole.
We connect them by paths of length 2 using pigeonhole



■

Now we can apply step 2 on $(G \setminus T) \cup \{u,v\}$

Step 3 Stitching up the pieces:

For every i, we take a different common neighbour of x_i and y_i in S.

Let $S_1 \subset S$ be the vertices of S used in this process.

We obtain a path P from u to v such that

$$V(G \setminus T) \subseteq V(P) \subseteq V(G \setminus T) \cup S_1$$

By Lemma 3, as T is an $(\{u, v\}, S)$ absorber, there exists a u-v path using exactly the vertices in $V(T) \setminus S_1$.

$\blacksquare$

References:
- Lecture notes of Alp Müyesser
- Talk of Matija Bucic
- Survey "Is laziness paying off" Szemerédi