The Absorption method

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Exercise 1.

Let n be an even integer and G be a n-vertex graph with minimum degree at least $n/2$. Using the switching method, prove that G has a perfect matching. Is it still true for smaller minimum degree ?

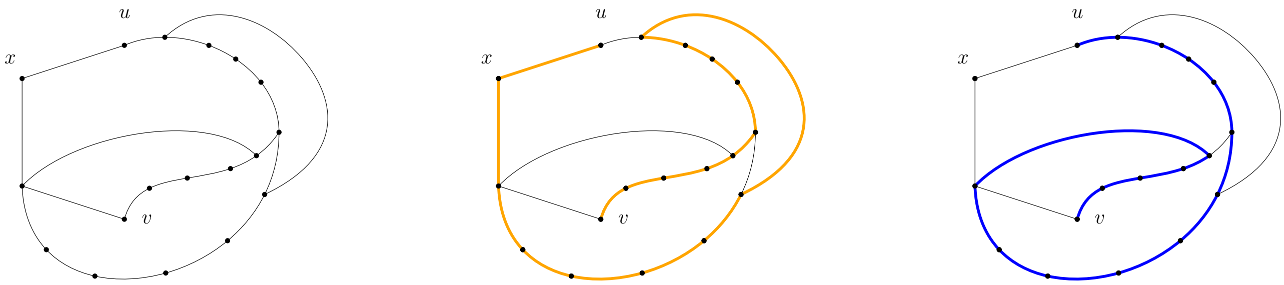
Exercise 2.

Recall from the lecture that an $(\{u,v\},S)$ absorber for Hamiltonian Paths is a graph T such that for any $S' \subseteq S$, there exists path P in T, with extremities u and v, such that $V(P) = V(T) \setminus S'$. We constructed such absorbers using triangles. In fact triangles are not necessary, we only need odd cycles.

Let $u, v, x \in V(G)$ and k and integer. Give a $(\{u,v\},x)$-absorber of girth k on $O(k)$ vertices.

Solution 2.

First solution:



Second solution:

Exercise 3. Erdős 1986

For every k, there exists N such that for every $n \geq N$ divisible by k, every n-vertex tournament can be partitioned into transitive n-vertex tournaments. What value of N does the proof of Erdős give?

Solution 3.

The absorber is composed of $k \cdot 2^{k-1}$ transitive tournaments on $2k-1$ vertices. To build it, we need that $N - (k \cdot 2^{k-1} - 1)(2k-1) \geq 2^{2k-2}$, i.e. $N = \Omega(4^k)$.

Exercise 4.

Let G be a graph of minimum degree $(\frac{1}{2} + \varepsilon)n$ and let S be a random subset of vertices of size C. Prove that with probability at least $1 - 4n^2 e^{-\varepsilon\Omega(C)}$, every pair of vertices in G has at least εC common neighbours in S.

Solution 4.

One can either first compute the number of common neighbours to u and v using Pigeon hole principle, before applying concentration and union bound, or first apply concentration and union bound before applying Pigeon hole.

For every u and v, let $X_{u,v} = N(u) \cap N(v) \cap S$. We have by pigeonhole principle $|N(u) \cap N(v)| \geq 2\varepsilon n$. We have by linearity of expectation $\mathbb{E}[X_{u,v}] = |N(u) \cap N(v)|C/n = 2\varepsilon C$. Let π be a random permutation of $[n] = V(G)$. Let S be the set of C vertices with largest $\pi(s)$. Permuting two elements of π affect $X_{u,v}$ by at most one. For every s, choosing the outcome of $\pi(w)$ on s common neighbours of u and v certifies that $X_{u,v} \geq s$. Hence, by McDiarmid,

$$\begin{aligned}\mathbb{P}(X_{u,v} \geq \varepsilon C) &\geq \mathbb{P}(|X_{u,v} - \mathbb{E}[X_{u,v}]| \geq \varepsilon C) \\ &\geq 1 - 4e^{-\varepsilon\Omega(C)}\end{aligned}$$

We then get that the result by union bound on all the couples of vertices.

The second option (which gives a better bound) is to first apply concentration. For every u, by McDiarmid,

$$\mathbb{P}(u \text{ has at least } (1+\varepsilon/2)C/2 \text{ neighbours in } S) \geq 1 - 4e^{-\Omega(C)}$$

So by union bound, every vertex of G has at least $(1+\varepsilon/2)C/2$ neighbours in S with probability at least $1 - 4ne^{-\Omega(C)}$. Assuming it is the case by Pigeon hole, every pair of vertices has at least εC common neighbours in C by Pigeon hole principle.

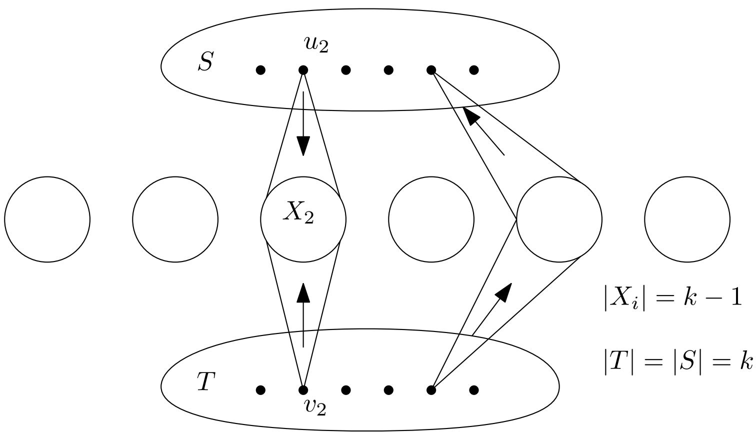
Exercise 5. Bucic and Sudakov 2023

Bucic and Sudakov improved the value of N in Exercise 3 to $N = \Omega(k^3 2^k)$. This exercise focuses on some of the absorbers they used. Let $\overrightarrow{G}$ be a tournament on $n \geq N$ vertices.
A set A of vertices is a *global absorber*, if for every subset $R \subset V(G) \setminus V(A)$ of less than 2^{k-1} vertices, $\overrightarrow{G}[A \cup R]$ can be tiled by transitive tournaments of size k. Let S be a set of k vertices. A set $L_S \subseteq V(G)$ is a *local absorber* for S if $\overrightarrow{G}[L_S]$ can be tiled by transitive tournaments of size k, $\overrightarrow{G}[L_{S,X} \cup S]$ can be tiled by transitive tournaments of size k.
Show that for any set S of k-vertices, there exists a local absorber L_S of size $O(k^2)$.

Note: Local absorbers are much weaker than global absorbers. Bucic and Sudakov handle the leftover with a reservoir and what remains of the reservoir with a collection of local absorbers.

Solution 5.

Let T be a transitive tournament on k vertices. Label $u_1, \dots u_k$ the vertices of S and $v_1, \dots v_k$ the vertices of T. For every i, there are four types of vertices: those that are inneighbours of both u_i and v_i, those that are outneighbours of both u_i and v_i, those that are inneighbours of u_i but outneighbours of v_i or vice-versa. By Pigeon-hole principle, one of those sets has size at least $n/4$. Denote if X_i. Find a disjoint collection of k transitive tournaments of size $k-1$, each in one of the X_i. Let $L_S = T \cup X_1 \cup \dots \cup X_k$. We claim that L_S is a local absorber for S: L_S can be tiled by the tournaments $(X_i \cup v_i)_i$ and $L_S \cup S$ can be tiled by the tournaments T and $(X_i \cup u_i)_i$.



★ Exercise 6.

The goal of this exercise is to provide a lower bound on the number of different perfect matchings in a graph of high minimum degree. Let $\varepsilon > 0$ and let G be a graph on n vertices with minimum degree $(1/2 + \varepsilon)n$.

Let $0 < \lambda$ an arbitrarily small constant and let C be a sufficiently large constant. Let S be a random subset of λn vertices of G, this will be our reservoir.

1. Prove that with probability $1 - o(1)$, every vertex u has at least $(1 + \varepsilon)|S|/2$ neighbours in S.

2. An S-absorber for perfect matchings is a graph T on an even number of vertices m, such that for every $S' \subseteq S$ of even size, $T \setminus S$ admits a perfect matching. Argue that the absorber constructed in the lecture for Hamiltonian paths also works here. Can you think of a simpler small absorber?

3. Partition randomly $V(G) \setminus V(T)$ into sets $V_1 \sqcup \ldots V_k$, each of size C. What proportion of sets V_i have $\delta(G[V_i]) \geq (1 + \varepsilon)C/2$?

4. Deduce that V_i contains a perfect matching for most i and absorb the leftover sets.

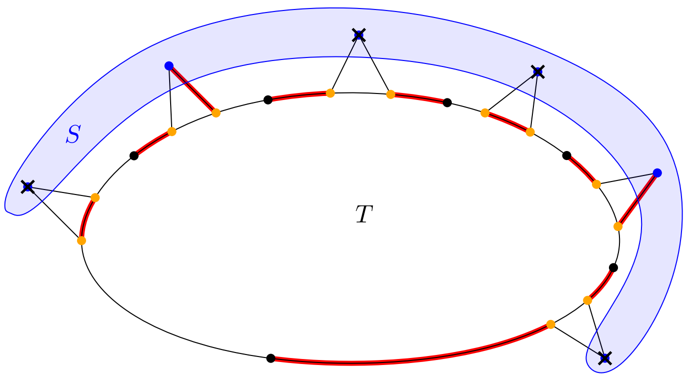
5. For every perfect matching M, upper bound the probability that the procedure described in the previous 4 steps produces M. Deduce that there are $\Omega(n)^{n/2}$ perfect matchings in G.

Note: Sárközy, Selkow, and Szemerédi proved in 2003 that every graph on n-vertices and minimum degree $\delta(G) \geq n/2$ has at least $\left(\frac{\delta(G)}{e+o(1)}\right)^n$ perfect matchings.

★ Solution 6.

1. This property is satisfied with probability at least $1 - 4ne^{-\Omega(|S|)} \geq 1 - o(1)$.

2. By using Pigeon hole principle repeatedly, one builds a collection of vertex disjoint triangles T_i, each containing exactly one vertex of S. Let u_i and v_i be the two vertices of $V(T_i) \setminus S$ (in orange on the next figure). By piegon hole principle, u_i and w_{i+1} have a common neighbour (in black).



In fact the reservoir S itself can here play the role of an absorber: in step 4, if W is sufficiently small compare to εS then matching W with vertices of S does not damage to much the minimum degree of the remaining vertices of S. Hence by Dirac theorem, what remains of S contains a perfect matching.

3. Like for the first item, the probability that a set V_i has $\delta(V_i) \leq (1+\varepsilon)C/2$ is at most $4Ce^{-\Omega(C)} \leq \lambda/100$ for C sufficiently large.

4. Let I be the set of indices i such that V_i satisfies $\delta(V_i) \geq (1 + \varepsilon)C/2$. For every $i \in I$, let M_i be an arbitrary perfect matching of $G[V_i]$ (such perfect matching exists by Exercise 1). Let $W = \bigcup_{i \notin I} V_i$. We have

$$\begin{aligned}
\mathbb{P}(|W| \geq |S|/10) &\leq \frac{10\mathbb{E}[|W|]}{|S|} && \text{by Markov's inequality} \\
&= \frac{C \cdot \mathbb{E}[|I|]}{\lambda n} && \text{by linearity of Expectation} \\
&\leq 1/10
\end{aligned}$$

If $|W| \leq \varepsilon|S|/50$, let S' be a subset of S of distinct neighbours of W (such set exists by the first item). Match each vertex $u \in W$ to its neighbour in S'. By the second item, $T - S'$ has a perfect matching.

5. This procedures produces a perfect matching with probability $9/10-o(1) \geq 1/2$. Let M be the random perfect matching produced by this algorithm and H be any fixed perfect matching.

$$\begin{aligned}\mathbb{P}(M = H) &\leq \mathbb{P}(M[V(G) \setminus (W \cup T)] = H[V(G) \setminus (W \cup T)] \\ &\leq \left(\frac{n}{2}\right)!\mathbb{P}(\forall xy \in E(H[V(G) \setminus (W \cup T)]),\ x \text{ and } y \text{ are in the same part } W_i) \\ &\leq \left(\frac{n}{2}\right)!\frac{(C!)^{\frac{n\lambda}{100C}}}{n!}\end{aligned}$$

The $\left(\frac{n}{2}\right)!$ accounts for the fact that the edges of H are unoriented. So the number of perfect matchings of G is at least $\frac{1}{2\max_H \mathbb{P}(M=H)} = \Omega(n)^n$