

The Absorption method

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Exercise 1.

Let n be an even integer and G be a n -vertex graph with minimum degree at least $n/2$. Using the switching method, prove that G has a perfect matching. Is it still true for smaller minimum degree?

Exercise 2.

Recall from the lecture that an $(\{u, v\}, S)$ absorber for Hamiltonian Paths is a graph T such that for any $S' \subseteq S$, there exists path P in T , with extremities u and v , such that $V(P) = V(T) \setminus S'$. We constructed such absorbers using triangles. In fact triangles are not necessary, we only need odd cycles.

Let $u, v, x \in V(G)$ and k and integer. Give a $(\{u, v\}, x)$ -absorber of girth k on $O(k)$ vertices.

Exercise 3. Erdős 1986

For every k , there exists N such that for every $n \geq N$ divisible by k , every n -vertex tournament can be partitioned into transitive n -vertex tournaments. What value of N does the proof of Erdős give?

Exercise 4.

Let G be a graph of minimum degree $(\frac{1}{2} + \varepsilon)n$ and let S be a random subset of vertices of size C . Prove that with probability at least $1 - 4n^2e^{-\varepsilon^2\Omega(C)}$, every pair of vertices in G has at least εC common neighbours in S .

Exercise 5. Bucic and Sudakov 2023

Bucic and Sudakov improved the value of N in Exercise 3 to $N = \Omega(k^3 2^k)$. This exercise focuses on some of the absorbers they used. Let \vec{G} be a tournament on $n \geq N$ vertices.

A set A of vertices is a *global absorber*, if for every subset $R \subset V(G) \setminus V(A)$ of less than 2^{k-1} vertices, $\vec{G}[A \cup R]$ can be tiled by transitive tournaments of size k . Let S be a set of k vertices. A set $L_S \subseteq V(G)$ is a *local absorber* for S if $\vec{G}[L_S]$ can be tiled by transitive tournaments of size k , $\vec{G}[L_S, X \cup S]$ can be tiled by transitive tournaments of size k .

Show that for any set S of k -vertices, there exists a local absorber L_S of size $O(k^2)$.

Note: Local absorbers are much weaker than global absorbers. Bucic and Sudakov handle the leftover with a reservoir and what remains of the reservoir with a collection of local absorbers.

★ Exercise 6.

The goal of this exercise is to provide a lower bound on the number of different perfect matchings in a graph of high minimum degree. Let $\varepsilon > 0$ and let G be a graph on n vertices with minimum degree $(1/2 + \varepsilon)n$.

Let $0 < \lambda$ an arbitrarily small constant and let C be a sufficiently large constant. Let S be a random subset of λn vertices of G , this will be our reservoir.

1. Prove that with probability $1 - o(1)$, every vertex u has at least $(1 + \varepsilon)|S|/2$ neighbours in S .
2. An S -absorber for perfect matchings is a graph T on an even number of vertices m , such that for every $S' \subseteq S$ of even size, $T \setminus S'$ admits a perfect matching. Argue that the absorber constructed in the lecture for Hamiltonian paths also works here. Can you think of a simpler small absorber?
3. Partition randomly $V(G) \setminus V(T)$ into sets $V_1 \sqcup \dots \sqcup V_k$, each of size C . What proportion of sets V_i have $\delta(G[V_i]) \geq (1 + \varepsilon)C/2$?

4. Deduce that V_i contains a perfect matching for most i and absorb the leftover sets.
5. For every perfect matching M , upper bound the probability that the procedure described in the previous 4 steps produces M . Deduce that there are $\Omega(n)^{n/2}$ perfect matchings in G .

Note: Sárközy, Selkow, and Szemerédi proved in 2003 that every graph on n -vertices and minimum degree $\delta(G) \geq n/2$ has at least $\left(\frac{\delta(G)}{e+o(1)}\right)^n$ perfect matchings.