$\boxed{\text{Second moment method 2}}$

1) Simple concentration Bound

Chernoff's Bound is limited to simple distributions. In practice it is often not enough.

Simple concentration Bound: Let $X = f(T_1, \dots T_n)$ where T_i are independant and such that

(*) Changing the outcome of one T_i affect X by at most c.

Then, $\mathbb{P}\left(|X - \mathbb{E}[X]| > t\right) \leq 2e^{-\frac{t^2}{2c^2 n}}$

$\rightsquigarrow$ For $X \sim Bin(n,p)$ we get $\leq 2e^{-\frac{t^2}{2n}}$ instead of $2e^{-\frac{t^2}{3np}}$

- which is very good for $p = \Theta(1)$ because S.C.B is much more general!
- For constant p, it is roughly the same.
- For $p = o(1)$ the simple concentration bound performs poorly:
  ex: $p = \sqrt{n}$ and $X \sim B(n,p)$

  C.B. $\mathbb{P}\left(|X - np| > \frac{1}{2} np\right) \leq 2e^{-\frac{\sqrt{n}}{16}}$

  S.C.B ______ $\leq 2e^{-1/16}$

- We often need $\mathbb{P}(|X - \mathbb{E}[X]| > \alpha \mathbb{E}[X]) \leq e^{-\beta \mathbb{E}[X]}$
  this works with S.C.B if $\mathbb{E}[X] = \Omega(n)$.

2) Talagrand's inequality (1995)

When $E[X] = o(n)$, we rather use Talagrand's inequality:

Talagrand's inequality 1: Let $X = f(T_1, \dots T_n) \geq 0$, such that there exist $c, r > 0$:

(i) Changing the outcome of one of the T_i affects X by at most c.

(ii) $\forall s$, if $X \geq s$, then there is a set of at most rs trials whose outcomes certify $X \geq s$.

Then for every $0 \leq t \leq \mathrm{Med}(X)$,

$$\mathbb{P}(|X - \mathrm{Med}(X)| > t) \leq 4 e^{-\frac{t^2}{8c^2 r \mathrm{Med}(X)}}$$

- Typically, c and r are small constants.
- Typically, $E[X]$ and $\mathrm{Med}(X)$ will be close: under the conditions of TI1, $|\mathrm{Med}(X) - E[X]| \leq 40c\sqrt{rE[X]}$

Talagrand's inequality 2: Under the same assumptions,

$$\mathbb{P}\left(|X - E[X]| > t + 60c\sqrt{r E[X]}\right) \leq 4 e^{-\frac{t^2}{8c^2 r E[X]}}$$

- Typically, if c and r are small constants and $t \gg \sqrt{E[X]}$.

Then, $\mathbb{P}(|X - E[X]| > t) \leq 2 e^{-\frac{\beta t^2}{E[X]}}$ $\quad \forall \beta < \frac{1}{8c^2 r}$.

- For $X \sim Bin(n,p)$, take $c = r = 1$.

$\rightarrow$ TI gives $\mathbb{P}(|X - np| > t) \leq 4e^{-\frac{t^2}{8np}}$ almost as good as C.B.

<u>Example</u>: Let G be a graph and H be a random subgraph of G: each edge of G is sampled independently with probability p.

$X = \#$ non-isolated vertices.

$\rightarrow$ Chernoff bound does not apply.

$\rightarrow$ $\mathbb{E}[X] \leq |V(G)|$ but the number of trials can be $\Omega(n^2)$ so S.C.B. performs poorly

$\rightarrow$ Taking $c = 2$ and $r = 1$, TI gives for every $t \gg \sqrt{|V(G)|}$

$$\mathbb{P}(|X - \mathbb{E}(X)| > t) \leq 2e^{-\frac{\beta t^2}{n}} \quad \text{for some } \beta > 0.$$

3) Application : Erdős - Szekeres :

[Erdős - Szekeres 35] Any permutation of $[n]$ contains a monotone subsequence of length at least $\sqrt{n}$.

Let σ be a random uniform permutation of $[n]$. Let X be the length of the longest increasing sequence of σ.

we have $\mathbb{E}[X] \approx 2\sqrt{n}$, but is X tightly concentrated?

[Frieze 1991] $P(|X - \mathbb{E}[X]| < \mathbb{E}[X]^{2/3}) \longrightarrow 1$

(weaker than our usual $\mathbb{E}[X]^{1/2}$).

[Talagrand 1995]: sample σ by sampling random reals in $[0,1]$. to have independance.

$$P\left(|X - \mathbb{E}[X]| > t + 60\sqrt{\mathbb{E}[X]}\right) < 4\, e^{-\frac{t^2}{8\mathbb{E}[X]}}.$$

Improved : [Baik, Deift, Johansson 1999] $\sqrt{\mathbb{E}[X]} \to \mathbb{E}(X)^{1/3}$

Exercise session ?

3) Colouring triangle-free graphs:

Theorem: There exists some $\Delta_0 \in \mathbb{N}$ s.t. for every triangle-free graph G of maximum degree $\Delta > \Delta_0$, $\chi(G) \leq \left(1 - \frac{1}{2e^6}\right)\Delta$

Proof: Let $C = \lfloor \frac{\Delta}{2} \rfloor$, we can assume that G is Δ-regular.

1. Assign to each vertex a random uniform colour in $[C]$.
2. If u is assigned the same colour as one of its neighbours, we uncolour u.
   Otherwise u retains its colour.
3. One by one colour every vertex that did not retain its colour by a free colour (not used in its neighbourhood).

Idea: After the first step, a good proportion of the vertices will have several neighbours that retained the same colour c, for many colours c.

For every vertex v, let X_v count the number of colours that are assigned to at least two neighbours of v and retained by all of them.

Let A_v be the event that $X_v \leq \frac{\Delta}{2e^6}$ and $\mathcal{E} = \{A_v : v \in V(G)\}$

A_v depends only on the colours of the vertices within a ball of radius 2.

So A_v is mutually independent from $\mathcal{E} \setminus \underbrace{\{A_u : \text{dist}(u,v) \leq 4\}}_{\# \leq \Delta^4}$

$\leadsto$ we need $P(A_v) \leq \frac{1}{4\Delta^4}$

Let $X_v' = \#$ colours c such that c is assigned to exactly two neighbours of v and retained by both of them.

Let $Y_{v,c} = \begin{cases} 1 & \text{if } \_\_\_\_\_\_\_\_ \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[X_v] \geq \mathbb{E}[X_v'] \underset{L.E}{=} \sum_{c \in [C]} \mathbb{E}[Y_{v,c}]$$

$$Y_{v,c} = 1 \text{ if } \exists u, w \in N(v) : \alpha(u) = \alpha(w) = c$$
$$\forall z \in \underbrace{N(u) \cup N(v) \cup N(w) - \{u,v,w\}}_{\# \leq 3\Delta - 3 \leq 6C}, \alpha(z) \neq c.$$

$$\mathbb{E}[X_v] \geq C\,\mathbb{E}[Y_{v,c}] \geq C \cdot \binom{\Delta}{2} \cdot \left(\frac{1}{C}\right)^2 \cdot \left(1 - \frac{1}{C}\right)^{6C}$$

$$e^{-x} < 1 - x + \frac{x^2}{2}$$

$$\geq \underbrace{C \frac{\Delta(\Delta-1)}{2C^2}} \cdot \left(e^{-\frac{1}{C}} - \frac{1}{2C^2}\right)^{6C}$$

$$\geq (\Delta - 1) \left(e^{-6} - \frac{e^{-\frac{5C-1}{C}} 6C}{2C^2} + o\left(\frac{1}{C^2}\right)\right)$$

$$\geq \frac{\Delta - 1}{e^6} \left(1 - \frac{3e}{C} + o\left(\frac{1}{C^2}\right)\right)$$

$$\geq \frac{\Delta}{e^6} - 1 \quad \text{for } C \text{ large}.$$

- Instead of proving the concentration of X_v, we work on two related variables.

Let $AT_v = \#$ colours assigned at least thrice in $N(v)$

Let Del_v = # colours assigned at least twice in $N(v)$ but deleted at least once.

We have $X_v = AT_v - Del_v$.

1. $\mathbb{P}(|AT_v - \mathbb{E}[AT_v]| > t) < 2e^{-\frac{t^2}{2\Delta}}$

AT_v depends on the trials of the Δ neighbors of v.
Changing one of these trials affect AT_v by at most $1 \rightsquigarrow c = 1$
we take $c = 1$ in the S.C.B

2. $\mathbb{P}(|Del_v - \mathbb{E}[Del_v]| > t) < 4e^{-\frac{t^2}{100\Delta}}$ For $t \geq \sqrt{\Delta \log \Delta}$

The value of Del_v depends on $O(\Delta^2)$ trials.
Changing one of these trials affect Del_v by at most $2 \rightsquigarrow c = 2$
$\mathbb{E}[Del_v] \leq \Delta \overset{!}{=} o(\Delta^2)$ so the S.C.B will perform poorly.

If $Del_v \geq s$, then there is a set of at most $3s$ trials that satisfy $Del_v \geq s$: for each such colour, two vertices in $N(v)$ + 1 that deletes it.

$\rightsquigarrow r = 3$.
We have $8c^2 r = 96 < 100$ and $t = \theta(\Delta) \gg \sqrt{\mathbb{E}[Del_v]}$ so by TI,

$$\mathbb{P}(|Del_v - \mathbb{E}[Del_v]| > t) < 2e^{-\frac{t^2}{100\,\mathbb{E}[Del_v]}}$$
$$< 2e^{-\frac{t^2}{100\Delta}}$$

3. $X_v = AT_v - Del_v$.

$$|X_v - \mathbb{E}[X_v]| = |AT_v - \mathbb{E}[AT_v] - (Del_v - \mathbb{E}[Del_v])|$$
$$\leq |AT_v - \mathbb{E}[AT_v]| + |Del_v - \mathbb{E}[Del_v]|$$

So $|X_v - \mathbb{E}[X_v]| > 2t \Rightarrow |AT_v - \mathbb{E}[AT_v]| > t$ or

$$|Del_v - \mathbb{E}(Del_v)| > t$$

So $\mathbb{P}(A_v) = \mathbb{P}\left(X_v \leq \frac{\Delta}{2e^6}\right)$

$$\leq \mathbb{P}\left(|X_v - \mathbb{E}[X_v]| > \frac{\Delta}{2e^6}\right)$$

$$< 2e^{-\frac{t^2}{2\Delta}} + 4e^{-\frac{t^2}{100\Delta}}$$ where $t = \frac{\Delta}{4e^6}$

$$< 5e^{-\frac{\Delta}{c}}$$ for some $C > 0$.

$$< \frac{1}{4\Delta^4}$$ for Δ large enough. ∎

$t \approx \sqrt{\Delta} \ln \Delta$
would work.

4) Azuma's inequality (1967)

Azuma's inequality: Let $X = f(T_1, \dots T_m)$ such that

(i) there exists $c_1, \dots c_m$ s.t. $\forall i, \forall t_1, \dots t_i, \forall t_i'$,

$$\left| \mathbb{E}[X \mid T_1 = t_1 \dots T_i = t_i] - \mathbb{E}[X \mid T_1 = t_1, \dots T_i = t_i'] \right| \leq c_i$$

then $\mathbb{P}(|X - \mathbb{E}[X]| > t) \leq 2 e^{-\frac{t^2}{2 \sum_i c_i^2}}$

Example: $m+1$ tosses of a coin and X count the number of coin tosses equal to the first one.

Observation: - Azuma is a "Martingale inequality".

- In the S.C.B., we first get all the random trial and then analyse how the modification of any of them affects X. there this modification is performed in the middle of the random experiment. So Azuma generalises S.C.B.

- Azuma can be applied to sequences of dependent random trials, so its much more general!

- In the S.C.B. the bound c is uniform here we bound by a function of $c_1, \dots c_m$, which often gives stronger results.

Bollobás 1988: $\chi(G(n, \frac{1}{2})) \sim \frac{n}{2 \log_2 n}$ almost always.

We already know that $\alpha(G(n, \frac{1}{2})) \sim 2 \log_2 n$ almost always

so $\chi(G(n, \frac{1}{2})) \geq \frac{n}{\alpha(G(n, \frac{1}{2}))} \gtrsim \frac{n}{2 \log_2 n}$ almost always

The reverse inequality was an open question for 25 years.

We recall that the expected number of k-cliques in $G(n, \frac{1}{2})$ is

$$f(n,k) = \binom{n}{k} 2^{-\binom{k}{2}}$$. Let $k \sim 2\log_2 n$ s.t. $f(n,k) > n^{3+o(1)}$

~~Proof~~ Lemma: $$\mathbb{P}(\omega(G) < k) < e^{-(c+o(1))(n^2/\ln^8 n)}$$

Proof: Consider an arbitrary ordering of the edges. Let $m = \binom{n}{2} \sim \frac{n^2}{2}$

Let G_0 be the empty graph on n vertices and

$$G_{i+1} = \begin{cases} G_i \cup e_{i+1} & \text{with probability } 1/2 \\ G_i \end{cases}$$

Let Y_i be the number of edge-disjoint k-cliques in G_i

→ Modifying one Y_i affects the subsequent Y_j by at most 1.

→ $\mathbb{E}[Y_m] \geq (1+o(1)) \frac{n^2}{2k^4}$ (We omit this).

So by Azuma,

$$\mathbb{P}\left(\omega(G(n,\tfrac{1}{2})) < k\right) = \mathbb{P}(Y_m = 0)$$

$$\leq \mathbb{P}\left(|Y_m - \mathbb{E}[Y_m]| \geq \mathbb{E}[Y_m]\right)$$

$$\leq e^{-\frac{\mathbb{E}[Y_m]^2}{2m}}$$

$$\leq e^{-(c'+o(1))\frac{n^2}{4k^8}}$$

$$\leq e^{-(c+o(1))\frac{n^2}{\ln^8(n)}}$$

∎

Let $l = \left\lfloor \frac{n}{\ln^2(n)} \right\rfloor$. For every set S of l vertices,

$G(n,\frac{1}{2})_{|S}$ has the distribution of $G(l,\frac{1}{2})$.

Let k such that $f(l,k) > n^{3+o(1)}$ and $k \sim 2\log_2 l \sim 2\log_2 n$

$$\mathbb{P}\left(\alpha(G_{|S}) < k \text{ for some } k\text{-sets}\right) \leq \binom{n}{m} e^{-(c+o(1))\frac{l^2}{\ln^8(l)}}$$

$$< 2^n e^{-(1+o(1))l^{3/2}}$$

$$\leq 2^{l^{1+o(1)}} e^{-(1+o(1))l^{3/2}}$$

$$= o(1)$$

So almost every l vertices contain a k-element independent set.

If G has this property, pick inductively k-element independent set until there are l vertices left. Use 1 colour for each independent set and 1 for every remaining vertex.

$$\chi(G) \leq \left\lceil \frac{n-l}{k} \right\rceil + l$$

$$\leq \frac{n}{k} + l$$

$$= \frac{n}{2\log_2 n}(1+o(1)) + o\left(\frac{n}{\log_2 n}\right)$$

$$= \frac{n}{2\log_2 n}(1+o(1)) \qquad \blacksquare$$

Note: We could have used Talagrand, but it didn't exist then.