# Improved concentration

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**Exercise 1.**
Show that Azuma's inequality implies the simple concentration bound.

**Exercise 2.**
We have three coins at our disposal: one is fair, the two others are biaised and land on head (resp. tail) with probability 2/3. Consider the following gambling game in $n$ rounds. At the first round, one tosses the fair coin. For every $i \geq 1$, if the $i$th round resulted in head (resp. tail), one tosses for the $i+1$th round the coin biaised towards head (resp. tail). If the coin landed on its edge, one tosses the fair coin, but we consider thatthis never happens.

Let $X_n$ be the number of heads obtained during this game. The winning of the player is $X_n - n$. Show that the game is fair and that $X_n$ concentrates.

**Solution 2.**
By symmetry, $\mathbb{E}[X] = n/2$ so the game is fair. Let $T_1, \dots T_n$ be the random experients corresponding to the coin tosses, where $T_i = 1$ if the $i$-th coin landed on head.

Let $x_i = \mathbb{E}\left[\sum_{j>n+1-i} T_j \middle| T_i = 1\right]$ and $y_i = \mathbb{E}\left[\sum_{j>n+1-i} T_j \middle| T_i = 0\right]$ For any $t_1, \dots t_{i-1}$, we have $x_i = \frac{2(x_{i-1}+1)+y_{i-1}}{3}$ and $y_i = \frac{x_{i-1}+1+2y_{i-1}}{3}$. Hence,

$$\begin{aligned}\mathbb{E}\left[X_n | T_1 = t_1, \dots T_{i-1} = t_{i-1}, T_i = 1\right] - \mathbb{E}\left[X_n | T_1 = t_1, \dots T_{i-1} = t_{i-1}, T_i = 0\right] &= x_i - y_i \\ &= \frac{x_{i-1} - y_{i-1}}{3} + 1/3 \\ &\leq 2/3 \leq 1\end{aligned}$$

So by Asuma's inequality, for every $t$, $\mathbb{P}(|X - n/2| \geq t) \leq 2e^{-\frac{t^2}{2n}}$

**Exercise 3.**
Let $c > 0$. Let $G$ be the graph on the vertex set $(\mathbb{Z}/7\mathbb{Z})^n$ with an edge between $u$ and $v$ if they differ on exactly one coordinate. Let $U$ be a set of $7^{n-1}$ vertices of $G$ and $W$ be the set of vertices of $G$ at distance more than $(c+2)\sqrt{n}$ form a vertex of $U$. Show that $|W| \leq 7^n e^{-c^2/2}$.

**Solution 3.**
Let $X = (T_1, \dots T_n)$ be a random vertex of $G$. Changing on of the coordinates of $X$ affects $\mathrm{dist}(X, U)$ by at most one. Hence, by the simple concentration bound,

$$\mathbb{P}(\mathrm{dist}(X,U) \leq \mathbb{E}[\mathrm{dist}(X,U)] - t) \leq e^{-\frac{t^2}{2n}}$$

By taking $t = 2\sqrt{n}$, we have $\mathbb{P}(\mathrm{dist}(X,U) \leq \mathbb{E}[\mathrm{dist}(X,U)] - t) \leq e^{-2} < 1/7$. As $\mathbb{P}(\mathrm{dist}(X,U) = 0) = |U|/|V(G)| = 1/7$, we have $\mathbb{E}[\mathrm{dist}(X,U)] \leq 2\sqrt{n}$.

By the simple concentration bound, we also have,

$$\begin{aligned}\mathbb{P}(\mathrm{dist}(X,U) \geq (c+2)\sqrt{n}) &\leq \mathbb{P}(\mathrm{dist}(X,U) \geq \mathbb{E}[\mathrm{dist}(X,U)] + c\sqrt{n}) \\ &\leq e^{-\frac{c^2}{2n}} = e^{-c^2/2}\end{aligned}$$

### Exercise 4. How tight is Erdős-Szekeres?

Let $\pi$ be a random uniform permutation of $[n]$. An increasing subsequence is a set of indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi(i_1) < \cdots < \pi(i_k)$. Let $X$ be the size of the largest increasing subsequence of $\pi$.

Admitting that $\mathbb{E}[X]$ is of order $2\sqrt{n}$, show that $X$ concentrates around its expected value.

### Solution 4.

Sample $\pi$ by choosing independently uniformly at random a real $y_i$ in $[0,1]$ for every $i \in [n]$ and let $\pi(i)$ be the order of $y_i$ within $\{y_i : i \in [n]\}$. with probability 1, the reals are distinct and this defines a random uniform permutation.

Modifying one $y_i$ affects $X$ by at most 1 and $X \geq r$ can be certified by $r$ variables $y_i$, so by Talagrand's inequality, there exists a constant $\beta > 1$ such that for every $n$ and $t \gg n^{1/4}$,

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq 2e^{-\frac{\beta t^2}{\sqrt{n}}}$$

### Exercise 5.

Let $G$ be a graph with chromatic number 1000. Let $U$ be a random subset of $V(G)$, such that each vertex $v$ belongs to $U$, independtly at random with probability $1/2$. Let $H$ be the graph induced by the vertices of $U$. Show that

$$\mathbb{P}[\chi(H) \leq 400] \leq 1/100.$$

### Solution 5.

Let $L$ be the graph induced by the complement of $U$. Consider two colourings of $L$ and $H$ using disjoint pallets of colours. The colouring of $G$ that restricts to these colourings is proper, so $\chi(G) \leq \chi(H) + \chi(L)$ Hence $\mathbb{E}[\chi(L)] \geq \chi(G)/2 = 500$.

Let $V_1 \sqcup \ldots V_{1000}$ be a 1000-colouring of $G$. Consider the random process where step $i$, we expose which of the vertices in $V_i$ belong to $U$. As $V_i$ is an independent set, all the vertices in $U \cap V_i$ can be coloured with one colour, hence, the conditional expected chromatic number of $L$ grows by at most one at each step. Hence by Azuma's inequality,

$$\begin{aligned}\mathbb{P}(\chi(L) \leq 400) &\leq e^{-\frac{100^2}{2000}} \\ &\leq e^{-5} < 1/100\end{aligned}$$

### ★ Exercise 6.

A graph is *$k$-choosable* if for every assignment $L : V(G) \to 2^{\mathbb{N}}$ of lists of colours to the vertices of $V(G)$, where $|L(v)| \geq k$ for every $v \in V(G)$, $G$ admits a proper colouring in which every vertex uses a colour from its list.

Show that there exists $\varepsilon > 0$ such that for every large enough $\Delta$, every triangle-free graph of maximum degree $\Delta$ is $(1-\varepsilon)\Delta$-choosable.

### ★ Solution 6.

The proof is highly similar to the one shown in the lecture to prove that triangle-free graphs are $(1 - \frac{1}{2e^6})\Delta$-colourable.