Improved concentration

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Exercise 1.

Show that Azuma's inequality implies the simple concentration bound.

Exercise 2.

We have three coins at our disposal: one is fair, the two others are biaised and land on head (resp. tail) with probability $2/3$. Consider the following gambling game in n rounds. At the first round, one tosses the fair coin. For every $i \geq 1$, if the ith round resulted in head (resp. tail), one tosses for the $i+1$th round the coin biaised towards head (resp. tail). If the coin landed on its edge, one tosses the fair coin, but we consider thatthis never happens.

Let X_n be the number of heads obtained during this game. The winning of the player is $X_n - n$. Show that the game is fair and that X_n concentrates.

Exercise 3.

Let $c > 0$. Let G be the graph on the vertex set $(\mathbb{Z}/7\mathbb{Z})^n$ with an edge between u and v if they differ on exactly one coordinate. Let U be a set of 7^{n-1} vertices of G and W be the set of vertices of G at distance more than $(c+2)\sqrt{n}$ form a vertex of U. Show that $|W| \leq 7^n e^{-c^2/2}$.

Exercise 4. How tight is Erdős-Szekeres?

Let π be a random uniform permutation of $[n]$. An increasing subsequence is a set of indices $1 \leq i_1 < \cdots < i_k \leq n$ such that $\pi(i_1) < \cdots < \pi(i_k)$. Let X be the size of the largest increasing subsequence of π.

Admitting that $\mathbb{E}[X]$ is of order $2\sqrt{n}$, show that X concentrates around its expected value.

Exercise 5.

Let G be a graph with chromatic number 1000. Let U be a random subset of $V(G)$, such that each vertex v belongs to U, independtly at random with probability $1/2$. Let H be the graph induced by the vertices of U. Show that

$$\mathbb{P}[\chi(H) \leq 400] \leq 1/100.$$

★ Exercise 6.

A graph is *k-choosable* if for every assignment $L : V(G) \to 2^{\mathbb{N}}$ of lists of colours to the vertices of $V(G)$, where $|L(v)| \geq k$ for every $v \in V(G)$, G admits a proper colouring in which every vertex uses a colour from its list.

Show that there exists $\varepsilon > 0$ such that for every large enough Δ, every triangle-free graph of maximum degree Δ is $(1-\varepsilon)\Delta$-choosable.