Random graphs

$p \ll f(n)$ means $p = f(n)$

1) Triangles in $G(n,p)$:

For which p does $G(n,p)$ contain a triangle with probability $1-o(1)$?

Theorem: If $p \ll \frac{1}{n}$, then $G(n,p)$ is triangle-free with probability $1-o(1)$.

Proof: $X = \#$ triangles in $G(n,p)$.

$$\mathbb{E}[X] = \binom{n}{3} p^3 \leq n^3 p^3 \qquad \text{by linearity of expectation}$$

By Markov's inequality, $\mathbb{P}(X \geq 1) \leq \mathbb{E}[X] = o(1)$. ■

When $p \gg \frac{1}{n}$, $\mathbb{E}[X] \to +\infty$ but maybe X is most of the time 0 and sometimes very big.

Theorem: If $p \gg \frac{1}{n}$, then $G(n,p)$ contains a triangle with probability $1-o(1)$.

Proof X_{uv}: indicator of $uv \in E(G)$

$$X_{uvw} = X_{uv} X_{uw} X_{vw} \quad \text{and} \quad X = \sum X_{uvw}$$

$$\mathrm{Cov}[X_{T_1}, X_{T_2}] = \mathbb{E}[X_{T_1} X_{T_2}] - \mathbb{E}[X_{T_1}]\mathbb{E}[X_{T_2}]$$
$$= p^{e(T_1 \cup T_2)} - p^{e(T_1)+e(T_2)} = \begin{cases} 0 & \text{if } |T_1 \cap T_2| \leq 1 \\ p^5 - p^6 & \text{---} = 2 \\ p^3 - p^6 & \text{---} = 3 \end{cases}$$

$$\#\{(T_i, T_j) : |T_i \cap T_j| = 2\} = O(n^4) \quad \text{so}$$

$$\operatorname{Var} X = \sum_{T_i, T_j} \operatorname{Cov}[X_{T_i}, X_{T_j}] = O(n^3)(p^3 - p^6) + O(n^4)(p^5 - p^6)$$

$$\lesssim n^3 p^3 + n^4 p^5 = o(n^6 p^6)$$

By Chebyshev's inequality,

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > \lambda \mathbb{E}[X]\right) \leq \frac{\operatorname{Var}(X)}{(\lambda \mathbb{E}[X])^2}$$

$$\leq \frac{o(n^6 p^6)}{\lambda^2 n^6 p^6} = o(1) \quad \blacksquare$$

Poisson limit: If $np \to c$ then X approaches a Poisson$(\lambda(c))$.

With method of moments if $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$ $\forall k \geq 0$ and X is nice, then $X_n^k \xrightarrow[\text{dist.}]{} X$

Asymptotic normality: We showed that $X \sim \mathbb{E}[X]$ w.h.p. in fact X satisfies the central limit theorem:

$$\frac{(X - \mathbb{E}[X])}{\sqrt{\operatorname{Var}(X)}} \xrightarrow[\text{dist}]{} \mathcal{N}(0,1)$$

(Method of moments Rucinski 1988).

Lemma: Let X s.t. $\mathbb{E}[X] \longrightarrow +\infty$.
If $\operatorname{Var}(X) = o(\mathbb{E}[X]^2)$ then $X \sim \mathbb{E}[X]$ almost always.

(Recall that $Y_n \longrightarrow Y$ almost always if $\mathbb{P}(\lim_{n \to +\infty} Y_n = Y) = 1$.)

Proof: By Chebyshev's inequality,

$$\mathbb{P}\left(|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]\right) \leq \frac{\operatorname{Var}[X]}{\varepsilon^2 \mathbb{E}[X]^2} \longrightarrow 0 \quad \blacksquare$$

2) A general proof method

Let $X = \sum X_i$ where $X_i = \mathbb{1}_{A_i}$ for some events A_i.

We say that $(X_1, \dots X_m)$ are symmetric if for every $i \neq j$, there exists a measure preserving mapping sending A_i to A_j.

$$\mathrm{Var}(X) = \sum_i \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j] \leq \mathbb{E}(X_i X_j) = \mathbb{P}(A_i \wedge A_j).$$

Let $I^* = \{ (i,j) : i<j \text{ and } A_i, A_j \text{ are not independent} \}$

Let $\Delta = \sum_{(i,j) \in I^*} P(A_i \wedge A_j)$. Because of

we have $\mathrm{Var}(X) \leq \mathbb{E}[X] + \Delta$

Corollary: If $X = \sum X_i$ of symmetric variables such that $\mathbb{E}[X] \longrightarrow +\infty$ and $\Delta = o(\mathbb{E}[X]^2)$ then $X > 0$ almost always and $X \sim \mathbb{E}[X]$ a.a.

$$\Delta = \sum_i \mathbb{P}(A_i) \underbrace{\sum_{(i,j) \in I^*} P(A_j | A_i)}_{\text{independent of } i \text{ if symmetry}} = \Delta^* \sum_i \mathbb{P}(A_i) = \Delta^* \mathbb{E}[X]$$

where $\Delta^* = \sum_{(i,j) \in I^*} \mathbb{P}(A_j | A_i)$. So $\mathrm{Var}(X) \leq (1 + \Delta^*) \mathbb{E}[X]$.

Lemma: If $X = \sum X_i$ of symmetric variables such that $\mathbb{E}[X] \longrightarrow +\infty$ and $\Delta^* = o(\mathbb{E}[X])$. Then $X > 0$ a.a. and $X \sim \mathbb{E}[X]$ almost always.

Proof $\mathrm{Var}[X] = (1 + \Delta^*) \mathbb{E}[X] = \mathbb{E}[X] + o(\mathbb{E}[X]^2)$ ■

Application to triangles:

Proof of theorem 2:

$$X = \sum X_{uvw} \quad \text{where} \quad X = \mathbb{1}_{A_{uvw}} = \begin{cases} 1 & \text{if } u \overset{w}{\triangle} v \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \binom{n}{3} p^3 \sim \frac{n^3 p^3}{6}.$$

$(X_{uvw})_{V(G)^3}$ is symmetric, $(u_1 v_1 w_1, u_2 v_2 w_2) \in I^*$ if they share 1 edge.

so $$\Delta^* = \sum_{\substack{T_2 \text{ sharing} \\ \text{an edge with} \\ T_1}} \mathbb{P}(A_{T_2} \mid A_{T_1}) \qquad \text{for some fixed } T_1$$

$$= (n-3) p^2 = o(\mathbb{E}[X]) \text{ because } p \gg \frac{1}{n}.$$

So by the previous lemma, $X \sim \mathbb{E}[X]$ a.a. ∎

2) Thresholds

Increasing / monotone property : $\forall A \subseteq B$, $P(A) \subset P(B)$

examples: • containing a subgraph H
• being connected
• not being k-colorable
• being Hamiltonian, having a perfect matching ..

A monotone property P has threshold $r(n)$ if

$$\mathbb{P}\left[P(G(n,p))\right] \longrightarrow \begin{cases} 0 & \text{if } p \ll r(n) \\ 1 & \text{if } p \gg r(n). \end{cases}$$

[Bollobás & Thomason 1987]: Every non trivial monotone property has a threshold.

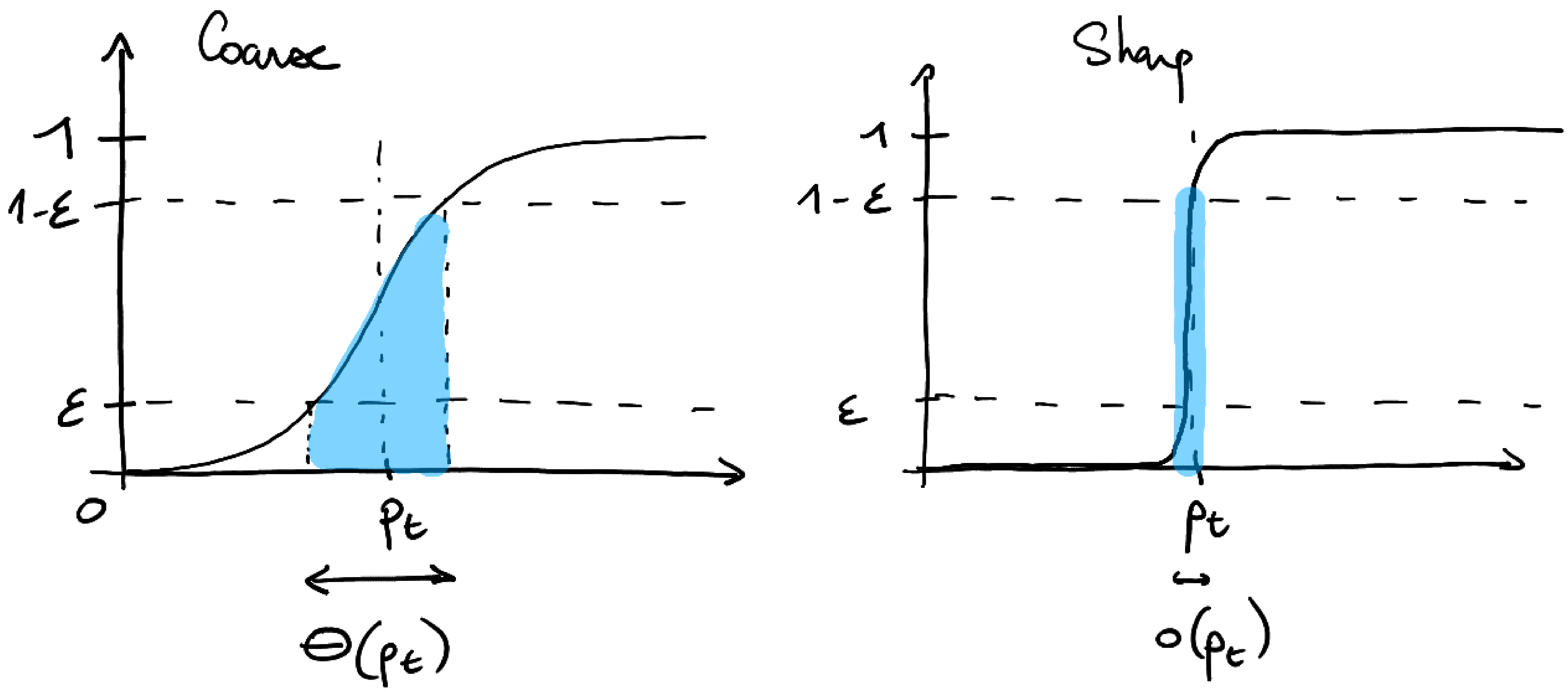
Note: the threshold function isn't uniquely defined. By abuse of language we often talk of the threshold by removing the negligeable terms / constant factors.

r_n is a sharp threshold for P if for every $\delta > 0$,

$$\mathbb{P}\left(P(G(n,p))\right) \longrightarrow \begin{cases} 0 & \text{if } p/r_n \leq 1-\delta \\ 1 & \text{if } p/r_n \geq 1+\delta \end{cases}$$

r_n is a coarse threshold for P if there exist $\varepsilon > 0$ and $0 < c < C$

$$\mathbb{P}\left(P(G(n,p))\right) \in [\varepsilon, 1-\varepsilon] \quad \text{whenever} \quad c \leq \frac{p}{r_n} \leq C$$



<u>Example</u>: $P(G(n,p)$ contains a triangle$) \longrightarrow 1 - e^{-c^3/6}$ if $np \to c$

[Friedgut 1999]: All monotone graph properties with a coarse threshold may be approximated by a local property.

For all monotone P with a coarse threshold, there exist a list $G_1, \dots G_m$ s.t. P is "close to" containing one of the G_i as a subgraph.

needed: e.g.
P is having a triangle and $\log_2 n$ edges is basically the same as containing a triangle.

Coarse	Sharp
Containing a subgraph H	Being connected / Positive mindegree
	Not being k-colorable
	Having a perfect matching / Hamiltonian

3) Containing a fixed subgraph:

Edge-vertex-ratio:
$$\rho(H) = \frac{|E(H)|}{|V(H)|}.$$

$$m(H) = \max_{H' \subseteq H} \rho(H')$$

Goldberg (1984) $m(H)$ can be computed in polynomial time.

Bollobás 1981: For every fixed graph H, $p = n^{-1/m(H)}$ is a threshold for containing H as a subgraph.

Subgraph H: $\boxed{\text{Coarse threshold at } n^{-1/m(H)}}$

Consequence: By Friedgut, coarse thresholds can only be of the form $n^{-\varepsilon}$.
A threshold of $\frac{\log n}{n}$ must be sharp

4) Connectivity and isolated vertices: $\boxed{\text{Sharp}}$

Let $p = \frac{\ln n + c_n}{n}$

$$\mathbb{P}(G(n,p) \text{ is connected}) \longrightarrow \begin{cases} 0 & \text{if } c_n \to +\infty \\ 1 - e^{e^{-c}} & \text{if } c_n \to c > 0 \\ 1 & \text{if } c_n \to 0 \end{cases}$$

Hitting time result

In fact, if you add the edges one at the time, then w.h.p, the graph becomes connected as soon as it has no isolated vertices.

The same holds for perfect matchings.

5) Clique number of $G(n,\frac{1}{2})$:

What is the clique number of $G(n,\frac{1}{2})$?

Let X_k be the number of k-cliques of $G(n,\frac{1}{2})$:

$$f(n,k) = \mathbb{E}[X_k] = \binom{n}{k} 2^{-\binom{k}{2}}$$

For what k is $f(n,k)$ small/large?

$$\boxed{\left(\frac{n}{ek}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k}$$

so $\quad \log_2(f(n,k)) = k\left(\log_2 n - \log_2 k - \frac{k}{2} + O(1)\right)$

Hence $f(k,n)$ transitions around $2\log_2 n$:

- if $k \geq (2+\delta)\log_2(n)$, $\quad f(k,n) \longrightarrow +\infty$
- if $k \leq (2-\delta)\log_2(n)$, $\quad f(k,n) \longrightarrow 0$

Theorem:
- if $f(k,n) \longrightarrow +\infty$ then $\omega(G(n,\frac{1}{2})) < k$ w.h.p
- if $f(k,n) \longrightarrow 0$, then $\omega(G(n,\frac{1}{2})) \geq k$ w.h.p

Proof: The first point follows from Markov's inequality as $\mathbb{E}[X_k] = o(1)$.

For the second point, given a set S of k vertices, let A_S be the probability that S induces a clique.

A_S and A_T are not independent if $|S \cap T| \geq 2$.

$$\Delta^* = \sum_{T : |S \cap T| \geq 2} \mathbb{P}(A_T \mid A_S)$$

$$= \sum_{i=2}^{k-1} \binom{k}{i} \binom{n-k}{k-i} 2^{\binom{i}{2} - \binom{k}{2}} \underset{\text{skip}}{=} o(\mathbb{E}[X])$$

So $X > 0$ w.h.p. ■

Two-point concentration for $\omega(G(n, 1/2))$

[Bollobás - Erdős 76 & Matula 76]

There exist $k(n)$, with $k(n) \sim 2 \log_2(n)$ such that $\omega(G(n, \frac{1}{2})) \in \{k, k+1\}$ w.h.p.

Proof: For $k \sim 2 \log_2(n)$,

$$\frac{f(n, k+1)}{f(n,k)} = \frac{\binom{n}{k+1} 2^{-\frac{k(k+1)}{2}}}{\binom{n}{k} 2^{-\frac{k(k-1)}{2}}} = \frac{n-k}{k+1} 2^{-k}$$

$$= n^{-1+o(1)}$$

Let $k_0 \sim 2 \log_2 n$ be the value such that

$$f(n, k_0) \geq n^{-1/2} > f(n, k_0+1)$$

Then $f(n, k_0 - 1) \longrightarrow +\infty$ and $f(n, k_0+1) = o(1)$.

so the clique number of $G(n, 1/2)$ is w.h.p in $\{k_0 - 1, k_0\}$. ■

Remarks: • Outside a sparse subset of integers n, we actually have a one point concentration.

• By taking the complement (which is also $G(n, \frac{1}{2})$), we obtain a two-point concentration of $\alpha(G(n, \frac{1}{2}))$.

[Bohman - Hofstad 2024] Two-point concentration of $\alpha(G(n,p))$ for all $p \geqslant n^{-2/3+\varepsilon}$

• Since $\chi(G) \geqslant \frac{n}{\alpha(G)}$, we have

$$\chi(G(n, 1/2)) \geqslant (1+o(1)) \frac{n}{2 \log_2 n} \quad \text{w.h.p.}$$

[Bollobás 1987] $\chi(G(n, 1/2)) \sim \frac{n}{2 \log_2 n}$ w.h.p.