Random graphs and thresholds

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Exercise 1.

Show that for every $\delta > 0$, a series of n independent coin flips contains k consecutive heads

- with probability $o(1)$ if $k \leq (1 + \delta)\log_2 n$,
- with probability $1 - o(1)$ if $k \leq (1 - \delta)\log_2 n$.

Exercise 2.

Let $S_{n,p}$ be a random subset of $[n]$ such that each element of $[n]$ is selected independently with probability p. For any fixed $k \geq 3$, determine the threshold at which $S_{n,p}$ contains an arithmetic progression of length k. Is it a a coarse or a sharp threshold?

Exercise 3. Threshold for cycles

What is the threshold for $G(n, p)$ to contain a cycle? Is it coarse or sharp?

Exercise 4. Poisson limit

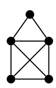
The k^{th} *moment* of a random variable X is $\mathbb{E}[X^k]$. A random variable X with finite moments is *determined by its moments* if for every random variable Y such that $\mathbb{E}[X^k] = \mathbb{E}[Y^k]$ for every k, then $X \overset{d}{=} Y$. In particular every random variable X such that there exists $C > 0$ such that $|\mathbb{E}[X^k]| \leq C^k k!$ for every k, is determined by its moments.
Let X be a random variable determined by its moments and let $(X_n)_{n \geq 0}$ be a sequence of random variables with finite moments such that for every k, $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$. Then X_n converges in distribution to X. Alternatively, all the previous definition hold by replacing moment by the k^{th} *factorial moment* $\mathbb{E}[X \cdot (X - 1) \cdots (X - k + 1)] = k! \cdot \mathbb{E}[\binom{X}{k}]$.
In particular Poisson distributions are determined by their moments and:

Theorem 1 *Let X_n be a sequence of random variables with finite moments, such that $\mathbb{E}[\binom{X_n}{k}] \to \lambda^k k!$ for every k. Then X_n converges in distribution to a random Poisson variable of parameter λ.*

Let $p(n) \sim c/n$ for some fixed constant $c > 0$. Let X_n be the random variable counting the number of triangles in $G(n, p)$.

1. Determine the asymptotics of $\mathbb{E}\left[\binom{X_n}{k}\right]$.
2. Let $\lambda > 0$ and $Y_n \sim \mathrm{Bin}(n, \lambda/n)$. Determine the asymptotics of of $\mathbb{E}\left[\binom{Y_n}{k}\right]$.
3. Let $m \in \mathbb{N}$, compute the limit of $\mathbb{P}(X_n = m)$ (we admit the fact that Y_n converges in distribution to the Poisson distribution of parameter λ). What is the asymptotic probability that $G(n, p)$ contains a triangle ?

Exercise 5.

Let H be the following graph:  Using an ad hoc proof, what is the threshold for having H as a subgraph?

Exercise 6.

Prove that there exists $c > 0$ such that $G(n, n^{-1/2})$ has asymptotically almost surely at least $cn^{3/2}$ edge-disjoint triangles.

★ Exercise 7. Threshold for connectedness

Let $p = \frac{\ln(n)+c_n}{n}$. The goal of this exercise is to determine the connectedness threshold of $G(n, p)$.

1. Prove that $\mathbb{P}[G(n,p) \text{ has no isolated vertex}] \to \begin{cases} 0 & \text{if } c_n \to -\infty \text{ arbitrarily slowly} \\ 1 & \text{if } c_n \to \infty \text{ arbitrarily slowly} \end{cases}$

2. By considering X_k count the number of connected components of size exactly k in $G(n, p)$, show that $\mathbb{P}[G(n,p)\text{is connected}] = 1 - o(1)$ if $c_n \to \infty$ arbitrarily slowly