Lovász Local Lemma

1) Principle and first example

Idea: Given a random procedure, let $\mathcal{E}$ be a set of "bad" events. If these events are independent, then the probability that the procedure succeeds is

$$\mathbb{P}\Big(\bigwedge_{A\in\mathcal{E}} \bar{A}\Big) = \prod_{A\in\mathcal{E}} \mathbb{P}(\bar{A}) > 0.$$

"If these events are not all independent, but almost, then this still works"

Lovász Local Lemma (Symmetric version).
Let $\mathcal{E}$ be a set of bad events such that for every $A\in\mathcal{E}$,
(1) $\mathbb{P}(A)\leq p<1$
(2) A is mutually independent of a set of all but at most d other events of $\mathcal{E}$.

If $\left|\begin{array}{l} 4pd\leq 1 \\ ep(d+1)\leq 1\end{array}\right.$, then the probability $\mathbb{P}\Big(\bigwedge_{A\in\mathcal{E}} \bar{A}\Big)$ that none of the events occur is positive.

⚠ This is false if we only have pairwise independence.

- A_e mutually independent from $\{A_f : f\in F\}$ means that A_e is independent from every $\{A_f : f\in F'\}$ with $F'\subset F$

Example: If $\mathcal{H}$ is a hypergraph such that each edge has size at least k and intersects at most $\frac{2^{k-1}}{e} - 1$ other edges. Then $\mathcal{H}$ is 2-colorable.

Proof: Consider a random uniform 2-coloring of the vertices. For each hyperedge e, let A_e be the event that e is monochromatic.

$$\forall e, \quad \mathbb{P}(A_e) \leq 2 \cdot 2^{-k} = 2^{-(k-1)}$$

Each event A_e is independent from $\{ f \in E(\mathcal{H}) : f \cap e = \emptyset \}$

There are at most $d = \frac{2^{k-1}}{e} - 1$ events A_f such that $e \cap f \neq \emptyset$.

As $e\, p\,(d+1) = 1$

by Lovász Local Lemma, there exists a proper 2-coloring of $\mathcal{H}$ ∎

Mutual independance Principle: Suppose that $\mathcal{X} = X_1, \dots X_m$ is a sequence of independent random experiments. Suppose that $A_1, \dots A_n$ is a series of events where each A_i is determined by $F_i \subseteq \mathcal{X}$. If $F_i \cap (F_{i_1}, \dots F_{i_k}) \neq \emptyset$ then A_i is mutually independent of $\{A_{i_1}, \dots A_{i_k}\}$.

Corollary: For $k \geq 9$, every k-regular k-uniform hypergraph is 2-colorable.

Proof: Every edge intersects intersects at most $d = k(k-1)$ other edges.

$$k(k-1) \leq \frac{2^{k-1}}{e} - 1 \qquad \text{for } k \geq 9. \quad ∎$$

False for $k=2$ (odd cycles), $k=3$ (Fano plane), true for $k \geq 4$ (Thomassen 92)

2) List colouring :

Theorem : Let L be a list assignment of colours to the vertices of a graph G. If for all $v \in V(G)$, $|L(v)| \geq \ell$ and each colour in $L(v)$ is used by at most $\frac{\ell}{8}$ neighbours of v, then G has a proper L-colouring.

<u>Proof</u> : Consider a random-uniform L-colouring.

For each edge e, for each colour i, let $A_{e,i}$ be the event that both endpoints of e are coloured i.

- For each e and i, $\mathbb{P}(A_{e,i}) \leq \frac{1}{\ell^2}$
- For each $e = uv$ and i, let $F_u = \{A_{f,j} \mid j \in L(u) \text{ and } f \text{ is incident to } u\}$

By the mutually independance principle, $A_{e,i}$ is independant from $\mathcal{E} - (F_u \cup F_v)$

$$|F_u| \leq \ell \cdot \frac{\ell}{8} \leq \frac{\ell^2}{8}$$

So $A_{e,i}$ is mutually independent for a set of all but $\frac{\ell^2}{4}$ events.

As $4 \frac{\ell^2}{4} \frac{1}{\ell^2} = 1$, by LLL, G admits a proper L-colouring ∎

Conjecture : $\ell/8$ can be replaced by $\ell - 1$
We would get $\ell/2e$ with the $ep(d+1)$ version.
Haxell 2001 : $\ell/2$ **Reed, Sudakov 2002** $\ell - o(\ell)$

3) Lower bound for Ramsey number

Recall that $R(s,t)$ is the minimum size of a clique K such that any 2-edge-colouring of K contains a blue K_s or a red K_t.

Theorem: If $e\binom{k}{2}\binom{n}{k-2}\cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$

Thus $\frac{\sqrt{2}}{e}(1+o(1))\, k\, 2^{k/2} < R(k,k)$

Proof: Consider a random 2-colouring of K_n.

Given a set S of k vertices, let A_S be the event that A_S is monochromatic.

$\mathbb{P}(A_S) = 2^{1-\binom{k}{2}}$ and A_S is mutually independent from $\{A_T : |T \cap S| \leq 1\}$.

Because $|T\cap S| \geq 2$ implies that S and T share an edge.

LLL with $d = \binom{k}{2}\binom{n-2}{k-2} - 1$ and $P(A_S) = 2^{1-\binom{k}{2}}$

because we shouldn't count S

gives the result ∎

The result is fine but LLL is not very powerful here

because $d > \binom{n}{k}^{1-O(1/k)}$ so there are a lot of dependencies.

It's much better for the off diagonal cases.

4) General LLL:

Theorem: Let $\mathcal{E}$ be a set of bad events and D be the dependancy graph on $\mathcal{E}$ such that:

(1) $\forall i$, A_i is mutually independent of all events $\{A_j : ij \notin E\}$.

(2) If there exist $x_1, \dots x_n \in [0,1[$ s.t.
$\forall i$, $\mathbb{P}(A_i) \leq x_i \prod_{ij \in E(D)} (1-x_j)$,

then $$\mathbb{P}\left(\bigwedge_{i=1}^{n} \bar{A}_i\right) \geq \prod_{i=1}^{n} (1-x_i).$$

Proof. We first prove by induction on s that for every $|S| \subseteq \{1, \dots n\}$, $|S| = s < n$, for every $i \in [n]$

$$\mathbb{P}\left(A_i \mid \bigwedge_{j \in S} \bar{A}_j\right) \leq x_i \qquad (*)$$

- True for $s = 0$.
- Assume that it holds for all $s' < s$.
  Let $S_1 = \{j \in S : ij \in E(D)\}$, $S_2 = S \setminus S_1$. Then

$$\mathbb{P}\left(A_i \mid \bigwedge_{j \in S} \bar{A}_j\right) = \frac{\mathbb{P}\left(A_i \wedge \left(\bigwedge_{S_1} \bar{A}_j\right) \mid \bigwedge_{S_2} \bar{A}_j\right)}{\mathbb{P}\left(\bigwedge_{S_1} \bar{A}_j \mid \bigwedge_{S_2} \bar{A}_j\right)}$$

→ Since A_i is mutually independent from $\{A_j : j \in S_2\}$,

$$\mathbb{P}\left(A_i \wedge \left(\bigwedge_{S_1} \bar{A}_j\right) \mid \bigwedge_{S_2} \bar{A}_j\right) \leq \mathbb{P}\left(A_i \mid \bigwedge_{S_2} \bar{A}_j\right) \qquad (**)$$

$$= \mathbb{P}(A_i) \leq x_i \prod_{ij \in E(D)} (1-x_j).$$

→ For the denominator, let $S_1 = \{j_1, \dots j_r\}$

If $r=0$, the denominator is 1 and $(*)$ holds.

Otherwise, $\mathbb{P}\left(\overline{A}_{j_1} \wedge \dots \wedge \overline{A}_{j_r} \,\middle|\, \bigwedge_{l\in S_2} \overline{A}_l\right)$

$$= \left(1 - \mathbb{P}\left(A_{j_1} \,\middle|\, \bigwedge_{j\in S_2} \overline{A}_l\right)\right) \cdots \left(1 - \mathbb{P}\left(A_{j_r} \,\middle|\, \overline{A}_{j_1} \wedge \dots \wedge \overline{A}_{j_{r-1}} \wedge \bigwedge_{l\in S_2} \overline{A}_l\right)\right)$$

$$\underset{\text{induction}}{\geq} (1-x_{j_1}) \cdots (1-x_{j_r}) \geq \prod_{ij\in D} (1-x_j)$$

$\rightarrow$ So $(*)$ holds and

$$\mathbb{P}\left(\bigwedge_{i=1}^{n} \overline{A}_i\right) = (1-\mathbb{P}(A_1)) \cdots \left(1 - \mathbb{P}\left(A_n \,\middle|\, \bigwedge_{i=1}^{n-1} \overline{A}_i\right)\right)$$

$$\geq \prod_{i=1}^{n} (1-x_i) \qquad \blacksquare$$

<u>Observation</u>: • The symmetric case follows by taking $x_i = \frac{1}{(d+1)}$ and using $\left(1-\frac{1}{d+1}\right)^d > \frac{1}{e}$.

• Lopsided Local Lemma: instead of asking for mutual independance, maybe avoiding some bad events makes it easier to avoid some other?

Replace (1) and (2) by

$\forall i, \forall S_2 \subseteq \{1,\dots n\} \setminus \{j : ij \in E(\mathcal{E})\}$,

$$\mathbb{P}\left(A_i \,\middle|\, \bigwedge_{j\in S_2} \overline{A}_j\right) \leq x_i \prod_{ij\in E} (1-x_j)$$

$\rightsquigarrow$ sufficient for $(**)$

, Application to Latin transversal Erdős & Spencer 1991

5) Application : Non-repetitive words.

Let Σ be an alphabet. A non-repetitive word on Σ is a word $w \in \Sigma^*$ such that w has no subword of the form xx.

Thue 1906: For every $n \geq 4$, there exist a non-repetitive word on Σ, if $|\Sigma| \geq 3$.

Let $sq(n)$ be the smallest k such that for every sequence $\Sigma_1, \dots \Sigma_n$ of alphabet of size k, there exists a non-repetitive word w where $w_i \in \Sigma_i \; \forall i$.

Theorem: $sq(n) \leq 8$.

Proof: Let $k \geq 8$.

Let $\Sigma_1, \dots \Sigma_n$ be a sequence of alphabets of size $\geq k$.

Let w be a random uniform word on $\Sigma_1, \dots \Sigma_n$.

Let $B_{i,\ell}$ be the event that $w_i \dots w_{i+2\ell-1}$ is a square.

$\forall i, \ell$, $\mathbb{P}(B_{i,\ell}) \leq k^{-\ell}$.

$B_{i\ell}$ is mutually independent from all other events but $D_{i\ell} = \{ B_{i'\ell'} : i \in [i-2\ell'+1, i+2\ell-1] \}$.

Let $x_\ell = x_{i,\ell} = \dfrac{C^m}{k^\ell + C^m}$ for some constant C to be adjusted later.

$$x_{i,\ell} \prod_{D_{i\ell}} (1 - x_{i',\ell'}) = x_\ell \prod_{m \geq 1} (1 - x_m)^{2(m+\ell-1)}$$

We need to prove that for some C and k,

$$\forall \ell \geq 1, \quad \frac{1}{k^\ell} \leq \frac{C^\ell}{k^\ell + C^\ell} \prod_{m \geq 1} \left(\frac{k^m}{C^m + k^m} \right)^{2(m+\ell-1)}$$

we have $1 + y \leq e^y$ so $\frac{k^m}{C^m + k^m} \geq \exp\left(-\frac{C^m}{k^m}\right)$

hence,

Let $\delta = \frac{c}{m}$

$$\prod_{m \geq 1} \left(\frac{k^m}{C^m + k^m} \right)^{2(m+\ell-1)} \geq \exp\left(-2 \sum_{m \geq 1} (m+\ell-1)\delta^m\right)$$

$$\geq \exp\left(-2C\left[(\ell-2)\sum_{m \geq 1} \delta^m + \sum_{m \geq 1} (m+1)\delta^m\right]\right.$$

$$\sum_{m \geq 1} \delta^m = \frac{\delta}{1-\delta} \quad \text{and} \quad \sum_{m \geq 1} (m+1)\delta^m = \sum_{m \geq 1} m\delta^{m-1} - 1$$

$$\sum_{p \geq 1} p\delta^{p-1} = \left(\sum_{p \geq 0} \delta^p\right)' = \left(\frac{1}{1-\delta}\right)' = \frac{1}{(1-\delta)^2}$$

so $$\prod_{m \geq 1} \left(\frac{k^m}{C^m + k^m} \right)^{2(m+\ell-1)} \geq \exp\left(-2\left(\frac{(\ell-2)\delta}{1-\delta} + \frac{1}{(1-\delta)^2} - 1\right)\right)$$

$$\geq \exp\left(-\frac{2\delta}{1-\delta}\ell + \frac{4\delta}{1-\delta} + \frac{1}{(1-\delta)^2} - 1\right)$$

$k = 16$, $C = 2$, $\delta = \frac{1}{8}$ $$\geq \exp\left(-\frac{2\ell}{7} + 1\right) \geq e$$

$\rightsquigarrow \forall \ell \quad \frac{1}{16^\ell} \leq \frac{2^\ell}{18^\ell} \cdot e \cdot e^{-\frac{2}{7}\ell}$

We could get better constants with a better analysis.

[Grytczuk, Przybyło, Zhu 2010] other version of LLL $\rightsquigarrow$ $k=4$