Lovász Local Lemma

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Exercise 1.

We recall that given a hypergraph $\mathcal{H}$ and a 2-colouring f of its vertices, the discrepancy of an edge e is the absolute value of the difference of number of red and blue vertices in e. The discrepancy of f is the maximum discrepancy over all edges e. The discrepancy of $\mathcal{H}$ is the minimum discrepancy of a 2-colouring of $\mathcal{H}$.

Let H be a k-uniform hypergraph in which each edge intersects at most d other edges. Show that if $d \le \frac{e^{\ell^2/6k}}{8}$ then H has discrepancy at most ℓ.

Solution 1.

We consider a random uniform 2-colouring of H. Consider the following set of bad events: for each edge e, let A_e be the event that e has discrepancy more than ell. Let X_e be the number of blue vertices in e. We have $\mathrm{disc}(e) = |2X_e - k|$. By Chernoff bound, we have

$$p = \mathbb{P}(A_e) \le 2e^{-\frac{\ell^2}{6k}}$$

Each A_e is mutually independent from all events but the A_f such that $e \cap f \neq \emptyset$. There are at most d such events. As $4pd \le 1$, the result follows from the symmetric LLL.

Exercise 2.

In the lecture, we proved the following result: Let G be a graph with a list assignment L such that for every vertex u, $|L(u)| \ge \ell$ and every colour in $L(u)$ belongs to at most $\ell/8$ lists of neighbours of u. Then G is L-colourable.

What would go wrong in our proof if we considered one of the following sets of bad events?

1. For each vertex v, A_v is the event that v has a neighbour identically coloured.
2. For each edge e, B_e is the event that e is monochromatic.

Solution 2.

Exercise 3. Alon and Linial 1989

Let D be a directed graph with minimum outdegree δ and maximum indegree Δ. Show that if $e(\Delta(\delta+1)+1)(1-1/k)^\delta \le 1$, then D has a directed cycle of length divisible by k.

Solution 3.

Without loss of generality we assume that each has outdegree precisely δ. Consider a random uniform k-colouring f of the vertices of D. For each v, let A_v be the event that v has no outneighbour coloured $f(v)+1 \mod k$. We have $\mathbb{P}(A_v) = \left(\frac{k-1}{k}\right)^\delta$. Each event A_v is mutually independent from $\{A_u \colon (\{u\} \cup N^+(u)) \cap (\{u\} \cup N^+(u)) = \emptyset\}$. That is, from all but $d = (\delta+1)\Delta$ events. Hence the symmetric LLL proves the existence of a k-colouring f such that each vertex v has an outneighbour coloured $f(v)+1 \mod k$. One can then find a directed cycle of length divisible by k by performing a walk increasing the colour by one at each step.

Exercise 4.

Let G be a graph of maximum degree Δ and $V_1 \sqcup \cdots \sqcup V_k$ be a partition of its vertices, with $|V_i| \geq 2e\Delta$ for every i. Prove that G has an idependent set with one vertex in each V_i.

Solution 4.

Up to removing some vertices from the V_i, we can assume that $|V_i| = \lceil 2e\Delta \rceil$ for every i. Pick independently uniformly at random one vertex in each V_i. Let S be the corresponding random set. For every edge e of G, let A_e be the event that both endpoints of e belong to S. We have $\mathbb{P}(A_e) \leq p = \frac{1}{\lceil 2e\Delta \rceil^2}$. Each event A_{uv} is mutually independent from all events but those of the form A_{xy} with x of y in the same part as u or v. There are $d \leq 2\Delta\lceil 2e\Delta \rceil - 1$ such events. As $ep(d+1) \leq \frac{2e\Delta}{\lceil 2e\Delta \rceil} \leq 1$, by LLL, with positive probablity none of the events A_e occur.

Exercise 5.

Prove that for every $\varepsilon > 0$, there exists an integer ℓ_0 such that for every $n > 0$, there exists a binary word $a_1 \dots a_n$ such that for every $\ell \geq \ell_0$ and every i, the subsequences $a_i, \dots a_{i+\ell-1}$ and $a_{i+\ell}, \dots a_{i+2\ell}$ differ on at least $(1/2 - \varepsilon)\ell$ coordinates.

Solution 5.

Let $\varepsilon > 0$, and N and ℓ_0 integers to be adjusted later. Sample each a_i uniformly independently at random. Let $A_{i,\ell}$ be the event that $a_i, \dots a_{i+\ell-1}$ and $a_{i+\ell}, \dots a_{i+2\ell}$ differ on less than $(1/2 - \varepsilon)\ell$ coordinates. By Chernoff's bound, $\mathbb{P}(A_{i,\ell}) \leq e^{-\frac{2(1/2-\varepsilon)^2\ell^2}{3\ell}} = C^\ell$ for some constant $C(\varepsilon) < 1$. Let D be the dependency graph such that $A_{i,\ell}$ and $A_{i',\ell'}$ are connected if $i' \in [i - 2\ell' + 1, i + 2\ell - 1]$. We finish the analysis with the assymmetric Lovasz Local Lemma as in the lecture for square free words.

★ Exercise 6. Latin transversals and the Lopsided Lovász Local Lemma

Let $A = (a_{ij})_{(i,j)\in[n]^2}$ be an array of integer entries. A Latin transversal is a permutation π of $[n]$ such that the entries $a_{i\pi(i)}$ are all distinct.

Prove using the Lopsided Lovász Local Lemma stated below, that if no integer appears more than $k \leq (n-1)/4e$ times in A, then A has a Latin transversal.
Hint: For every tuple of indices $(i, j, i', j') \in [n]^4$ such that $i < i'$, $j \neq j'$ and $a_{ij} = a_{i'j'}$, consider the event that $\pi(i) = j$ and $\pi(i') = j'$.

★ Solution 6.

Let π be a random uniform permutation of $[n]$. Let T be the set of all tuples of indices $(i, j, i', j') \in [n]^4$ such that $i < i'$, $j \neq j'$ and $a_{ij} = a_{i'j'}$. For every $t = (i, j, i', j') \in T$, consider the event A_t that $\pi(i) = j$ and $\pi(i') = j'$. Note that π is a Latin transversal if none of the events A_t is verified. We have $\mathbb{P}(A_t) \leq \frac{1}{n(n-1)} = p$. Consider the following graph D on T. For every $t_1 = (i_1, j_1, i'_1, j'_1)$ and $t_2 = (i_2, j_2, i'_2, j'_2)$ in T, we connect t_1 with t_2 if $\{i_1, i'_1\} \cap \{i_2, i'_2\} \neq \emptyset$ or $\{j_1, j'_1\} \cap \{j_2, j'_2\} \neq \emptyset$. In other words, t_1 and t_2 are such that (i_1, j_1), (i'_1, j'_1), (i_2, j_2) and (i'_2, j'_2) occupy distinct rows and columns. The maximum degree of G is at most $d = 4nk - 1$: for every $(i_1, j_1, i'_1, j'_1) \in T$, there are $2n$ possible choices of coordinate on the same line or column as (i_1, j_1) or (i'_1, j'_1), and at most k other coordinate with the same entry, minus one for t_1 itself. We have $ep(d+1) \leq 1$. Hence it suffices to show that D is a negative correlation graph to conclude by the Symmetric Lovász Local Lemma.

I haven't finished typing this correction.

Definition 1 (Negative dependency graph). *Let $A_1, \dots A_n$ be a collection of events. A directed graph G on the vertex set $[n]$ is a negative dependency graph every A_i is positively correlated with its non neighbours: for every $i \in [n]$, for every $S \subset [n] \setminus N^+(i)$,*

$$\mathbb{P}\left(A_i \;\middle|\; \bigwedge_{j\in S} \overline{A_j}\right) \leq \mathbb{P}(A_i).$$

Theorem 1 (Symmetric Lopsided Lovász Local Lemma). *Let $A_1, \dots A_m$ be a collection of events in an arbitrary probability space and D be a negative correlation graph on the events $(A_i)_{i\in[n]}$. Let d be the maximum degree of D and $p = \max_i \mathbb{P}(A_i)$. If $ep(d+1) \leq 1$, then, $\mathbb{P}\left(\bigwedge_{i\in[n]} \overline{A_i}\right) > 0$.*

Theorem 2 (Asymmetric Lopsided Lovász Local Lemma)**.** *Let $A_1, \dots A_m$ be a collection of events in an arbitrary probability space and D be a negative correlation graph on the events $(A_i)_{i\in[n]}$. Suppose that there exists a collection of reals $x_i \in [0, 1)$ such that for every i,*

$$\mathbb{P}(A_i) \le x_i \prod_{j\in N^+(i)} (1 - x_j).$$

Then, $\mathbb{P}\left(\bigwedge_{i\in[n]} \overline{A_i}\right) \ge \prod_{i\in[n]}(1 - x_i)$.