Lovász Local Lemma

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Exercise 1.

We recall that given a hypergraph $\mathcal{H}$ and a 2-colouring f of its vertices, the discrepancy of an edge e is the absolute value of the difference of number of red and blue vertices in e. The discrepancy of f is the maximum discrepancy over all edges e. The discrepancy of $\mathcal{H}$ is the minimum discrepancy of a 2-colouring of $\mathcal{H}$.

Let H be a k-uniform hypergraph in which each edge intersects at most d other edges. Show that if $d \le \frac{e^{\ell^2/6k}}{8}$ then H has discrepancy at most ℓ.

Exercise 2.

In the lecture, we proved the following result: Let G be a graph with a list assignment L such that for every vertex u, $|L(u)| \ge \ell$ and every colour in $L(u)$ belongs to at most $\ell/8$ lists of neighbours of u. Then G is L-colourable.

What would go wrong in our proof if we considered one of the following sets of bad events?

1. For each vertex v, A_v is the event that v has a neighbour identically coloured.
2. For each edge e, B_e is the event that e is monochromatic.

Exercise 3. Alon and Linial 1989

Let D be a directed graph with minimum outdegree δ and maximum indegree Δ. Show that if $e(\Delta(\delta+1)+1)(1-1/k)^{\delta} \le 1$, then D has a directed cycle of length divisible by k.

Exercise 4.

Let G be a graph of maximum degree Δ and $V_1 \sqcup \cdots \sqcup V_k$ be a partition of its vertices, with $|V_i| \ge 2e\Delta$ for every i. Prove that G has an idependent set with one vertex in each V_i.

Exercise 5.

Prove that for every $\varepsilon > 0$, there exists an integer ℓ_0 such that for every $n > 0$, there exists a binary word $a_1 \dots a_n$ such that for every $\ell \ge \ell_0$ and every i, the subsequences $a_i, \dots a_{i+\ell-1}$ and $a_{i+\ell}, \dots a_{i+2\ell}$ differ on at least $(1/2-\varepsilon)\ell$ coordinates.

★ Exercise 6. Latin transversals and the Lopsided Lovász Local Lemma

Let $A = (a_{ij})_{(i,j)\in[n]^2}$ be an array of integer entries. A Latin transversal is a permutation π of $[n]$ such that the entries $a_{i\pi(i)}$ are all distinct.

Prove using the Lopsided Lovász Local Lemma stated below, that if no integer appears more than $k \le (n-1)/4e$ times in A, then A has a Latin transversal.
Hint: For every tuple of indices $(i,j,i',j') \in [n]^4$ such that $i < i'$, $j \ne j'$ and $a_{ij} = a_{i'j'}$, consider the event that $\pi(i) = j$ and $\pi(i') = j'$.

Definition 1 (Negative dependency graph). *Let $A_1, \dots A_n$ be a collection of events. A directed graph G on the vertex set $[n]$ is a negative dependency graph every A_i is positively correlated with its non neighbours: for every $i \in [n]$, for every $S \subset [n] \setminus N^+(i)$,*

$$\mathbb{P}\left(A_i \;\middle|\; \bigwedge_{j\in S} \overline{A_j}\right) \le \mathbb{P}(A_i).$$

Theorem 1 (Symmetric Lopsided Lovász Local Lemma)**.** *Let* $A_1, \dots A_m$ *be a collection of events in an arbitrary probability space and* D *be a negative correlation graph on the events* $(A_i)_{i\in[n]}$*. Let* d *be the maximum degree of* D *and* $p = \max_i \mathbb{P}(A_i)$*. If* $ep(d+1) \leq 1$*, then,* $\mathbb{P}\left(\bigwedge_{i\in[n]} \overline{A_i}\right) > 0$*.*

Theorem 2 (Asymmetric Lopsided Lovász Local Lemma)**.** *Let* $A_1, \dots A_m$ *be a collection of events in an arbitrary probability space and* D *be a negative correlation graph on the events* $(A_i)_{i\in[n]}$*. Suppose that there exists a collection of reals* $x_i \in [0,1)$ *such that for every* i*,*

$$\mathbb{P}(A_i) \leq x_i \prod_{j\in N^+(i)} (1 - x_j).$$

Then, $\mathbb{P}\left(\bigwedge_{i\in[n]} \overline{A_i}\right) \geq \prod_{i\in[n]}(1 - x_i)$*.*