Second Moment method

March 12, 2026

Exercise 1.

Prove that with probability $1 - o(1)$, every bipartite subgraph of $G(n, 1/2)$ has at most $n^2/8 + n^{3/2}$ edges.

Solution 1.

Let A and B be two disjoint set of vertices. Let X be the number of edges induces by $A \sqcup B$ in $G \sim G(n, 1/2)$. We have $X \sim \text{Bin}(|A||B|, 1/2)$. Hence the expected number of edges in H is $|A| \cdot |B|/2 \le n^2/8$ and by Chernoff bound,

$$\begin{aligned} \mathbb{P}(X \ge n^2/8 + n^{3/2}) &\le \mathbb{P}(X \ge \mathbb{E}(X) + n^{3/2}) \\ &\le 2e^{-\frac{n^3}{3n/2}} \le 2e^{-\frac{2n^2}{3}} = o(2^{-n}) \end{aligned}$$

We conclude by union bound on the sets (A, B).

Exercise 2.

Given a hypergraph $\mathcal{H}$ and a 2-colouring f of its vertices, the discrepancy of an edge e is the absolute value of the difference of number of red and blue vertices in e. The discrepancy of f is the maximum discrepancy over all edges e. The discrepancy of $\mathcal{H}$ is the minimum discrepancy of a 2-colouring of $\mathcal{H}$.

Let $\mathcal{H}$ be a k-uniform hypergraph with k edges. Show that $\mathcal{H}$ has discrepancy less than $\sqrt{8k \ln k}$.

Solution 2.

Consider a random 2-colouring of $\mathcal{H}$. For every edge e, the number X_e of blue edges follows a binomial distribution of parameter k and $1/2$. The discrepancy of e is $|2X_e - k|$. By Chernoff bound, we have

$$\mathbb{P}\left(|X_e - k/2| \ge \sqrt{2k\ln(k)}\right) \le 2e^{-\frac{4\ln(k)}{3}} \le 2k^{-4/3} < k$$

So by linearity of Expectation, the expected number of edges with discrepancy at least $\sqrt{8k \ln k}$ is less than 1 and by First moment method, $\mathcal{H}$ admits a 2-colouring with discrepancy less than $\sqrt{8k \ln k}$.

Exercise 3.

Prove that there exists balanced bipartite graphs on $2n$ vertices with $\Omega(n^{4/3})$ edges but no $K_{2,2}$.

Solution 3.

Let G be the random balanced bipartite graphs on $2n$ vertices such that each edge appears independently with probability $n^{-2/3}$. The expectation of the number of $K_{2,2}$ is G is less than $n^{4/3}/4$. So by Markov's inequality, G contains at most $n^{4/3}/2$ copies of $K_{2,2}$ with probability greater than $1/2$. The number E of edges in G is follows a binomial law $B(n^2, n^{-4/3})$. So

$$\begin{aligned} \mathbb{P}[E > 3n^{4/3}/4] &\ge \exp\left(-\frac{(n^{4/3}/4)^2}{3n^{1-4/3}}\right) \qquad \text{by Chernoff bound} \\ &\ge \exp\left(-\frac{n^2}{48}\right) \end{aligned}$$

So for n large enough, by Markov's inequality, G has at least $n^{4/3}/3$ edges with probability more than $1/2$. Hence, with positive probability, G has at least $3n^{4/3}/4$ edges *and* at most $n^{4/3}/2$ copies of $K_{2,2}$. By removing one arbitrary edge in each of those copies, one obtains a graph with at least $n^{4/3}/4$ edges and no $K_{2,2}$.

Exercise 4.

Show that for every $\delta > 0$, a series of n independent coin flips contains k consecutive heads

- with probability $o(1)$ if $k \geq (1+\delta)\log_2 n$,
- with probability $1 - o(1)$ if $k \leq (1-\delta)\log_2 n$.

Solution 4.

Let $X_1, \dots X_n$ be n independent Bernoulli variable of parameter $1/2$. Let H_k be the number of sequences of k consecutive 1 in $(X_1, \dots X_n)$. That is, $H_k = \sum_{i=1}^{n-k} X_{i+0} \cdots X_{i+k-1}$. We have $\mathbb{E}(H_k) = (n-k+1)2^{-k}$ by linearity of expectation. We also have

$$\begin{aligned}
\mathrm{Var}(H_k) &= \sum_{i=1}^{n-k+1} \mathrm{Var}(X_{i+0} \cdots X_{i+k-1}) + \sum_{j=1}^{k-1} \sum_{i=1}^{n-k-j} \mathrm{Cov}(X_{i+0} \cdots X_{i+k-1}, X_{i+j+0} \cdots X_{i+j+k-1}) \\
&\leq n2^{-k}(1-2^{-k}) + n\sum_{j=1}^{k-1}(2^{-(2k-j)} - 2^{-2k}) \\
&\leq n2^{-k} + n2^{-2k}\sum_{j=1}^{k-1}(2^j - 1) \\
&\leq n2^{-(k-1)}
\end{aligned}$$

- If $k \geq (1+\delta)\log_2 n$, then $\mathbb{E}(H_k) \leq 2n^{-\delta} = o(1)$ and by Markov's inequality, $\mathbb{P}(H_k > 0) = o(1)$
- If $k \leq (1-\delta)\log_2 n$, then $\mathbb{E}(H_k) \geq n^{\delta}/2 \to \infty$ and $\mathrm{Var}(H_k) \leq n^{\delta}$.

$$\begin{aligned}
\mathbb{P}(H_k \leq \mathbb{E}(H_k)/2) &\leq \frac{\mathrm{Var}(H_k)}{(\mathbb{E}(H_k)/2)^2} && \text{By Chebyshev's inequality} \\
&\leq 4n^{-\delta} = o(1)
\end{aligned}$$

So $\mathbb{P}(H_k > 0) = 1 - o(1)$.

Exercise 5.

Prove that there exists $c > 1$ such that for all n, there exists c^n point in $\mathbb{R}^n$ with the property that every triple of points forms a triangle with angles at most 61°.

Solution 5.

Let $\varepsilon > 0$ such that any triangle with sides of length in $[1-\varepsilon, 1+\varepsilon]$ has all its angles of degree at most 61°.

Let $x_1, \dots x_m$ be m independent random uniform points in $\{-1,1\}^n$. for every i and j, let X_{ij} be the number of coordinates on which x_i and x_j differ, we have $X_{ij} \sim \mathrm{Bin}(n, 1/2)$. The distance between x_i and x_j is $\sqrt{2X_{ij}}$. By Chernoff bound,

$$\begin{aligned}
\mathbb{P}\left(\frac{\mathrm{dist}(x_i,x_j)}{\sqrt{n}} \in [1-\varepsilon, 1+\varepsilon]\right) &\geq \mathbb{P}\left(\frac{\mathrm{dist}(x_i,x_j)}{\sqrt{n}} \in [\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}]\right) \\
&\geq 1 - \mathbb{P}\left(|\mathrm{dist}(x_i,x_j) - n| > \varepsilon\right) \\
&\geq 1 - 2e^{-2n\varepsilon^2/3}
\end{aligned}$$

Let N be the number of pairs $\{i,j\}$ such that $|\frac{\mathrm{dist}(x_i,x_j)}{\sqrt{n}} - 1| > \varepsilon$. By linearity of expectation, we have $\mathbb{E}[N] \leq m^2 e^{-2n\varepsilon^2/3}$, which is less than 1 if $m = (e^{\varepsilon^2/3})^n$. We conclude by the fist moment method.

Exercise 6.

Let $(x_1, y_1), \dots (x_n, y_n)$ be n points of $\mathbb{Z}^2$ such that for all i, we have $|x_i|$, $|y_i| \leq \frac{2^{n/2}}{100\sqrt{n}}$. Show that there exists two disjoint sets of vertices I and J, not both empty, such that $\sum_{i\in I}(x_i, y_i) = \sum_{j\in J}(x_j, y_j)$.

Solution 6.

Let I be a random uniform subset of $[n]$. Let $X = \sum_{i\in I} x_i$ and $Y = \sum_{i\in I} y_i$. We have $\mathrm{Var}(X) = \sum_{i\in[n]} x_i^2/4 \leq \frac{2^n}{40000}$. So,

$$\begin{aligned}\mathbb{P}\left(|X - \mathbb{E}[X]| > \frac{2^{n/2}}{100}\right) &\leq \frac{2^n}{40000} \cdot \left(\frac{2^{n/2}}{100}\right)^{-2} && \text{by Chebyshev's inequality}\\ &\leq \frac{1}{4}\end{aligned}$$

By subadditivity of probability, $\mathbb{P}\left(|X| \text{ and } |Y| \leq \frac{2^{n/2}}{100}\right) \geq 1/2$. Thus, there are at least 2^{n-1} sets I such that (X, Y) belongs to the square S centered in the origin of side $\frac{2^{n/2}}{100}$. As $S \cap \mathbb{Z}^2$ contains $\frac{2^n}{10000}$, by pigeon hole principle, there exists $I \neq J$ such that $\sum_{i\in I}(x_i, y_i) = \sum_{j\in I}(x_j, y_j)$? We conclude by taking $I' = I \setminus J$ and $J' = J \setminus I$.

Exercise 7.

Let X be a set of integers, k an integer and p a prime such that $|\{x \mod p \colon x \in X\}| \geq 4k^2$. Prove that there exists an integer a such that $\{ax \mod p \colon x \in X\}$ intersects every interval of length at least p/k in $[p-1]$.

Solution 7.

Denote $X = \{x_1, \dots x_n\}$. Pick independently uniformly at random a and b in $[0, p-1]$. Let u be an integer (to be adjusted later). Let Y_i the indicator variable that $(ax + b \mod p) \in [0, u-1]$. Let $Y = \sum_i Y_i$. We have

$$\begin{aligned}\mathbb{E}[Y] &= \sum_i \mathbb{E}[Y_i] && \text{by linearity of Expectation}\\ &= \frac{nu}{p}\end{aligned}$$

For every $(c_i, c_j) \in [p-1]^2$, the following system has a unique solution:

$$\begin{cases} ax_i + b \equiv c_i \\ ax_j + b \equiv c_j \end{cases} \Leftrightarrow \begin{cases} a \equiv (x_i - x_j)^{-1}(c_i - c_j) \text{ and } b \equiv c_i - ax_i & \text{if } c_i \neq c_j \\ a \equiv 0 \text{ and } b \equiv c_i & \text{if } c_i = c_j \end{cases}$$

As a result $ax_i + b$ is independent from $ax_j + b$ and $\mathrm{Var}[Y] = \sum_i \mathrm{Var}[Y_i] = n\frac{u(p-u)}{p^2} < \mathbb{E}[Y]$. By Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}(Y = 0) &\leq \mathbb{P}(|Y - \mathbb{E}[Y]| \geq \mathbb{E}[Y])\\ &\leq \frac{\mathrm{Var}(Y)}{\mathbb{E}[Y]^2}\\ &< \frac{1}{\mathbb{E}[Y]} = \frac{p}{nu}\end{aligned}$$

Hence, there exists $a' \in [0, p-1]$ such that $\mathbb{P}(aX + b \in [0, u-1] | a = a') < \frac{p}{nu}$. In other words, $a'X$ intersects all but less than p^2/nu intervals. By taking $n \geq 4k^2$ and $u = \lceil p/2k \rceil$, $a'X$ intersects all but less than $\frac{p^2}{4k^2\lceil p/2k\rceil} \leq \frac{p^2}{4k^2 p/2k} \leq \frac{p}{2k} \leq u$ intervals of the form $[b', b' + u - 1]$. But $a'X \cap [b', b' + \lceil p/k \rceil - 1]$ if and only if $a'X$ misses all intervals $[b', b' + u - 1], \dots [b' + \lceil p/k \rceil - u, b' + \lceil p/k \rceil - 1]$. There are $\lceil p/k \rceil - \lceil p/2k \rceil = \lfloor p/2k \rfloor \geq u$ such intervals, a contradiction.

★ Exercise 8.

Prove that for some $c > 0$, the following holds. For every n, for every positive reals $a_1 \geq \cdots \geq a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i^2 = 1$, if $(\varepsilon_1, \ldots \varepsilon_n)$ is a random uniform vector of $\{-1, 1\}^n$, then

$$\mathbb{P}\left(\left|\sum_{i=1}^n \varepsilon_i a_i\right| \leq 1\right) \geq c.$$

★ Solution 8.

Hint:

1. Assuming that $a_1 \geq 1/2$, we prove that $\mathbb{P}\left(\left|\sum_{i=1}^n \varepsilon_i a_i\right| \leq 1\right) \geq 1/3$.
2. Assuming that $a_i < 1/2$ for all i, we show that there exists a bipartition (A, B) such that the sums $\sum_{i \in A} a_i^2$ and $\sum_{i \in B} a_i^2$ belong to $[3/8, 5/8]$. Conclude.

Without loss of generality, we assume that all a_i are positive and ordered in non-increasing order. Let $S = \left|\sum_{i=1}^n \varepsilon_i a_i\right|$.

1. Assume that $a_1 \geq 1/2$. Let $Y = \sum_{i=2}^n \varepsilon_i a_i$. We have

$$\begin{aligned} \mathrm{Var}[Y] &= \sum_{i=2}^n \mathrm{Var}[X_i] && \text{because the } X_i \text{ are independent} \\ &= \sum_{i=2}^n a_i^2 = 1 - a_1^2 \end{aligned}$$

Hence by Chebyshev's inequality, $\mathbb{P}(|Y| \geq 1+a_1) \leq \frac{1-a_1^2}{(1+a_1)^2} \leq \frac{1-a_1}{1+a_1} \leq 1/3$. Hence $\mathbb{P}(|Y| \leq 1+a_1) \geq 2/3$ and by symmetry of Y, $\mathbb{P}(Y \in [0, 1+a_1]) = \mathbb{P}(Y \in [-1-a_1, 0]) \geq 1/3$. We have

$$\begin{aligned} \mathbb{P}(|S| \leq 1) \geq\ & \mathbb{P}(X_1 = a_1 \text{ and } Y \in [-1-a_1, 0]) \\ & + \mathbb{P}(X_1 = -a_1 \text{ and } Y \in [0, 1+a_1]) \\ \geq\ & 1/6 + 1/6 = 1/3 && \text{by independance of } X_1 \text{ and } Y \end{aligned}$$

2. We now assume that $a_i \leq 1/2$ for all i. As $a_i^2 \leq 1/4$ for every i, changing one i form A to B modifies each the sum by at most $1/4$, hence there exists a bipartition (A, B) such that the sums $\sum_{i \in A} a_i^2$ and $\sum_{i \in B} a_i^2$ belong to $[3/8, 5/8]$. Let $S_1 = \sum_{i \in A} \varepsilon_i a_i$ and $S_2 = \sum_{i \in B} \varepsilon_i a_i$. We have $\mathbb{E}[S_1] = 0$ by symmetry and $\mathrm{Var}[S_1] = \sum_{i \in A} a_i^2 \in [3/8, 5/8]$. By Chebyshev's inequality, $\mathbb{P}(|S_1| \geq 1) \leq 5/8$, so $\mathbb{P}(|S_1| \leq 1) \leq 3/8$. Here again, by symmetry, $\mathbb{P}(S_1 \in [0, 1]) = \mathbb{P}(S_1 \in [-1, 0]) \leq 3/16$. Likewise, $\mathbb{P}(S_2 \in [0, 1]) = \mathbb{P}(S_2 \in [-1, 0]) \leq 3/16$. Hence,

$$\begin{aligned} \mathbb{P}(|S| \leq 1) \geq\ & \mathbb{P}(S_1 \in [0, 1] \text{ and } S_2 \in [-1, 0]) \\ & + \mathbb{P}(S_1 \in [-1, 0] \text{ and } S_2 \in [0, 1]) \\ \geq\ & 2 \cdot \left(\frac{3}{16}\right)^2 = 9/128 \end{aligned}$$