

Second Moment method

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Exercise 1.

Prove that with probability $1 - o(1)$, every bipartite subgraph of $G(n, 1/2)$ has at most $n^2/8 + n^{3/2}$ edges.

Exercise 2.

Given a hypergraph \mathcal{H} and a 2-colouring f of its vertices, the discrepancy of an edge e is the absolute value of the difference of number of red and blue vertices in e . The discrepancy of f is the maximum discrepancy over all edges e . The discrepancy of \mathcal{H} is the minimum discrepancy of a 2-colouring of \mathcal{H} .

Let \mathcal{H} be a k -uniform hypergraph with k edges. Show that \mathcal{H} has discrepancy less than $\sqrt{8k \ln k}$.

Exercise 3.

Prove that there exists balanced bipartite graphs on $2n$ vertices with $\Omega(n^{4/3})$ edges but no $K_{2,2}$.

Exercise 4.

Show that for every $\delta > 0$, a series of n independent coin flips contains k consecutive heads

- with probability $o(1)$ if $k \geq (1 + \delta) \log_2 n$,
- with probability $1 - o(1)$ if $k \leq (1 - \delta) \log_2 n$.

Exercise 5.

Prove that there exists $c > 1$ such that for all n , there exists c^n point in \mathbb{R}^n with the property that every triple of points forms a triangle with angles at most 61° .

Exercise 6.

Let $(x_1, y_1), \dots, (x_n, y_n)$ be n points of \mathbb{Z}^2 such that for all i , we have $|x_i|, |y_i| \leq \frac{2^{n/2}}{100\sqrt{n}}$. Show that there exists two disjoint sets of vertices I and J , not both empty, such that $\sum_{i \in I} (x_i, y_i) = \sum_{j \in J} (x_j, y_j)$.

Exercise 7.

Let X be a set of integers, k an integer and p a prime such that $|\{x \bmod p : x \in X\}| \geq 4k^2$. Prove that there exists an integer a such that $\{ax \bmod p : x \in X\}$ intersects every interval of length at least p/k in $[p - 1]$.

★ Exercise 8.

Prove that for some $c > 0$, the following holds. For every n , for every positive reals $a_1 \geq \dots \geq a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i^2 = 1$, if $(\varepsilon_1, \dots, \varepsilon_n)$ is a random uniform vector of $\{-1, 1\}^n$, then

$$\mathbb{P} \left(\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 1 \right) \geq c.$$