I/ Introduction :

Modalities: Attendance to exercise session is mandatory.
- 50% final exam with "cheat sheet"
- 25% Exercises
- 25% mid-term test

1) History :

Pioneered by Paul Erdős in the 50's.
Counter-example to many conjectures in conferences

Pros and cons of the probabilistic method :
- Allows to consider large unstructured graphs
  - $\rightarrow$ graphs untractable by computer search
  - $\rightarrow$ Any explicit construction introduces structure
    ex: most graphs have $\omega(G) = O(\log n) = \alpha(G)$
    it took a long time to explicit anything better than $o(\sqrt{n})$.
- Easy to use
  - $\rightarrow$ ex: average number of cliques of size k :
    $\mathbb{P}(S \text{ is a clique}) = 2^{-\binom{k}{2}}$
    $\mathbb{E}[\# \text{ clique of size } k] = \sum_{|S|=k} 2^{-\binom{k}{2}} = \binom{n}{k} 2^{-\binom{k}{2}}$
    linearity of expectation

(Maybe do this later).

- Most graphs are counterexamples
- Non constructive

II/ First moment method

First moment Principle: If $\mathbb{E}[X] \leq t$ then $\mathbb{P}(X \leq t) > 0$

Linearity of Expectation $\mathbb{E}\left[\sum_{i=1}^{\ell} X_i\right] = \sum_{i=1}^{\ell} \mathbb{E}[X_i]$

Markov's Inequality $\mathbb{P}(X \geq t) \leq \dfrac{\mathbb{E}[X]}{t}$

The linearity of Expectation allows to estimate $\mathbb{E}[X]$ easily, often by double counting.

Markov's Inequality is particularly useful if X is integer-valued and $\mathbb{E}[X] < 1$, we then have

$$0 \underset{FM}{<} \mathbb{P}(X=0) \quad \text{But} \quad \mathbb{P}(X=0) = 1 - \mathbb{P}(X \geq 1) \underset{MI}{\geq} 1 - \mathbb{E}[X]$$

General idea: count the number X of witnesses of P and show $\mathbb{E}[X] < 1$.

1) 2-colouring of Hypergraphs

Definition of hypergraph, colouring, uniform, property B ↗ Felix Bernstein 1908

[Erdős 63] If $\mathcal{H}$ has less than 2^{k-1} hyperedges, each of size at least k, then $\mathcal{H}$ is 2-colourable!

Proof: Random 2-colouring.

$$X_e = \begin{cases} 1 & \text{if } e \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases} \qquad X = \sum_{e \in E(\mathcal{H})} X_e .$$

$$\forall e, \quad \mathbb{E}[X_e] = \mathbb{P}(e \text{ is monochromatic}) \leq 2\, 2^{-k} \leq 2^{-(k-1)}$$

So $$\mathbb{E}[X] \underset{LE}{=} \sum_{e \in E(\mathcal{H})} \mathbb{E}[X_e] \leq |E(\mathcal{H})| \cdot 2^{-(k-1)} < 1.$$

so $\mathbb{P}(X=0) \underset{FM}{>} 0$ ■

Alternative proof: (without First moment)

$A_e = \{ e \text{ is monochromatic} \}$.

$$\mathbb{P}\Big(\bigcap_{e\in E(\mathcal{H})} \overline{A_e}\Big) = 1 - \mathbb{P}\Big(\bigcup_{e\in E(\mathcal{H})} A_e\Big)$$

$$\mathbb{P}\Big(\bigcup_{e\in E(\mathcal{H})} A_e\Big) \underset{\text{subadditivity}}{\leq} \sum_e \mathbb{P}(A_e)$$

$$\leq |E(\mathcal{H})|\, 2^{-(k-1)} < 1. \quad \blacksquare$$

$\boxed{\text{Same calculations, and subadditivity is a special case of FM}}$

$m(k) = \min \{ |E(\mathcal{H})| : \mathcal{H} \text{ k-uniform not 2-colorable} \}$.

We proved [Erdős 63] $\quad m(k) \geq 2^{k-1}$

[Radhakrishnan & Srinivasan 2000] $\quad m(k) \geq \Omega\left(\left(\frac{k}{\ln(k)}\right)^{1/2} 2^k\right)$

[Erdős 64] $\quad m(k) < (1+o(1)) \dfrac{e \ln(2)}{4} k^2 2^k$

(Give them as guided hard exercises).

2) Diagonal Ramsey numbers

Ramsey number $R(s,t) = \min\left\{ n : \begin{array}{l} \text{any 2-coloring of } K_n \text{ has} \\ \text{a blue } K_s \text{ or red } K_t \end{array} \right\}$

[Erdős 47] If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. So

$$R(k,k) > \lfloor 2^{k/2} \rfloor \quad \forall k \geq 3.$$

Proof: Consider a random 2-coloring of K_n.

For every $S \subset V(K_n)$ of size k,

$$X_S = \begin{cases} 1 & \text{if } S \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases} \quad \text{and } X = \sum X_S$$

$$\mathbb{P}(S \text{ is monochromatic}) = \mathbb{E}[X_S] = 2 \cdot 2^{-\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

$$\mathbb{E}[X] \underset{\text{L.E.}}{=} \binom{n}{k} 2^{1-\binom{k}{2}} < 1, \text{ so } \mathbb{P}(X=0) \underset{\text{F.M.}}{>} 0$$

There is a 2-coloring with no monochromatic clique of size k
so $R(k,k) > n$.

For $k \geq 3$, take $n = \lfloor 2^{k/2} \rfloor$

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \frac{2^{k/2+1}}{2^{k^2/2}} < \frac{2}{k!} < 1$$

■

3) Triangle-free graphs with high χ

> Sometimes we need to start with a random object and modify it slightly

[Erdős 59] $\forall k$, there exists a Δ-free graph with $\chi(G) \geq \Delta$

Proof $G \sim G_{n,p}$ where $p = n^{-2/3}$

Sketch: 1. G has no large independent set $P(\alpha < \frac{n}{2k}) > 0$ $\quad \chi > \frac{n}{\alpha}$
2. G has not too many triangles.
3. Delete the triangles

1. Let $S \subseteq V(G)$ of size $\lceil \frac{n}{2k} \rceil$

$$I_S = \begin{cases} 1 & \text{if } S \text{ is a stable set} \\ 0 & \text{if } S \text{ induces an edge} \end{cases} \qquad I = \sum_{|S| = \lceil \frac{n}{2k} \rceil} I_S .$$

$$\mathbb{E}[I_S] = \mathbb{P}(S \text{ is a stable set}) = (1-p)^{\binom{\lceil \frac{n}{2k} \rceil}{2}}$$

$$\boxed{\binom{n}{k} \leq 2^n} \qquad \boxed{(1+x) \leq e^x}$$

$$\mathbb{E}[I] \underset{\text{L.E.}}{=} \binom{n}{\lceil \frac{n}{2k} \rceil} (1-p)^{\binom{\lceil n/(2k) \rceil}{2}}$$

$$\leq 2^n \cdot \left(e^{-p}\right)^{\frac{n(n-2k)}{2 \cdot 4k^2}}$$

$$\leq 2^n \cdot e^{-n^{-2/3} \frac{n^2}{16k^2}} \quad \text{for } n \text{ large enough.}$$

$$\leq 2^n \cdot e^{-\frac{n^{4/3}}{16k^2}} \xrightarrow[n \to +\infty]{} 0$$

$$< \frac{1}{2} \quad \text{for } n \text{ large enough}$$

So $\mathbb{P}(I > 0) \underset{MI}{<} \frac{1}{2}$ for n large enough

$\rightsquigarrow$ G has no large independent set so χ must be big!

2. Unfortunately, G has triangles BUT not many of them

$T =$ # triangles.

$$\mathbb{E}[T] = \binom{n}{3} p^3 \leq \frac{n^3}{3!} \left(n^{-2/3}\right)^3 = \frac{n}{6}$$

So $\mathbb{P}\left(T \geq \frac{n}{2}\right) \underset{M.I}{<} \frac{1}{3}$

3. Since $\mathbb{P}(T \geq \frac{n}{2}) + \mathbb{P}(I > 0) < 1$,

$$\mathbb{P}(T < \tfrac{n}{2} \text{ and } I = 0) > 0.$$

We choose a set of $\frac{n}{2}$ vertices, hitting all triangles. removing it gives G', and cannot increase α.

so $\chi(G') \geq \dfrac{|V(G')|}{\alpha(G')} > \dfrac{n}{2\left\lceil \frac{n}{2k} \right\rceil} > k$. ■

In fact there exist graphs with arbitrarily large χ and girth

$\rightsquigarrow$ locally looks like a tree but high χ.

4) Erdős-Ko-Rado

$\mathcal{F}$ is intersecting if $\forall A, B \in \mathcal{F}$, $A \cap B \neq \emptyset$.

[Erdős-Ko-Rado 38] Let $\mathcal{F}$ be an intersecting family of k-element subsets of $[n]$, with $n \geq 2k$. Then,

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Proof by Katona 1972

Lemma For $s \in [n]$, let $A_s = \{s, s+1, \ldots, s+k-1\}$ mod n. Then $\mathcal{F}$ contains at most k of the sets A_s.

Proof Fix $A_s \in \mathcal{F}$. $A_t \cap A_s \Leftrightarrow |t-s| \leq k-1$

The set of A_t intersecting A_s can be partitioned into $k-1$ pairs (A_{s-i}, A_{s+k-i}).

$\mathcal{F}$ contains at most one element of each pair. $\square$

Let σ be a random uniform permutation of $[n]$,
i be a random uniform element of $[n]$.

Let $A = \{\sigma(i), \sigma(i+1), \ldots \sigma(i+k-1)\}$.

$$\mathbb{P}(A \in \mathcal{F} \mid \sigma \text{ fixed}) = \mathbb{P}(A_i \in \sigma^{-1}(\mathcal{F}) \mid \sigma \text{ fixed})$$

$$\leq \frac{k}{n} \quad \text{by the lemma}$$

so $\mathbb{P}(A \in \mathcal{F}) \leq \frac{k}{n}$.

$$X_A = \begin{cases} 1 & \text{if } A \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases} \qquad |\mathcal{F}| = \sum X_A.$$

$$|\mathcal{F}| = \sum_{\subseteq \mathcal{F}} \mathbb{E}[X_A]$$

so $$|\mathcal{F}| = \binom{n}{k} \mathbb{P}(A \in \mathcal{F}) \leq \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1} \quad \blacksquare$$

5) Dominating set

<u>Theorem</u>: Let G be an n-vertex graph with minimum degree $\delta > 1$. Then G has a dominating set of size at most

$$n \, \frac{1 + \ln(\delta+1)}{\delta+1} \quad \text{vertices}$$

<u>Proof</u>: Let $p \in [0,1]$ to be adjusted later.

Let S be a random subset of vertices, where $v \in S$ with probability p, independently.

Let $X = V(G) \setminus N(S)$.

$\mathbb{E}[S] = np$ by linearity of expectation.

$\forall v \in V(G)$, $\mathbb{P}(v \in X) = (1-p)^{|N(v)|+1} \leq (1-p)^{\delta+1}$.

So $\mathbb{E}[|X|] \leq n(1-p)^{\delta+1}$.

Hence $\mathbb{E}[|S| + |X|] = n(1-p)^{\delta+1} + np \leq n\left(e^{-p(\delta+1)} + p\right)$

$\boxed{\text{We now optimize } p}$

Let $f : p \mapsto n\left(e^{-p(\delta+1)} + p\right)$. $f'(p) = -(\delta+1)e^{-p(\delta+1)} + 1$

$$f'(p) = 0 \iff e^{-p(\delta+1)} = \frac{1}{\delta+1}$$

$$\iff p = \frac{\ln(\delta+1)}{\delta+1}.$$

By substituting, $\mathbb{E}[|S| + |X|] \leq n \, \dfrac{1 + \ln(\delta+1)}{\delta+1}$

So $\mathbb{P}(|S| + |X| \leq \_\_\_) > 0$ ■

FM

6) Better bound on Ramsey number

Previously we proved $R(k,k) > \lfloor 2^{k/2} \rfloor$

Actually a more careful analysis would have given

$$R(k,k) > \frac{1}{e\sqrt{2}} (1+o(1))\, k\, 2^{k/2}$$

Theorem For all n, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$

Hence, $R(k,k) > \frac{1}{e}(1+o(1))\, k\, 2^{k/2}$

Proof : 1. Consider a random uniform edge colouring of K_n.

X = # monochromatic cliques of size k

$$\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}} = m.$$

By F.M.M., there exists a colouring with at most m monochrom. cliques.

We delete one vertex in each of these cliques.

2. Maximise $n - \binom{n}{k} 2^{1-\binom{k}{2}}$

Taking $n \sim \frac{1}{e}(1+o(1))\, k\, 2^{k/2}$,

we have $\binom{n}{k} 2^{1-\binom{k}{2}} \ll n$ so $R(k,k) > \frac{(1+o(1))}{e} k\, 2^{k/2}$