First Moment method

March 5, 2026

Exercise 1.

Let G be a graph with a matching on m edges. Prove that G has a bipartite subgraph H with at least $\frac{1}{2}(|E(G)| + m)$ edges.

Exercise 2. Graphs with large chromatic number and girth

Prove that for every integers k and g, there exists a graph with chromatic number at least k and girth at least g.

Exercise 3. List colouring bipartite graphs

Let G be an n-vertex bipartite graph. Let $L : V(G) \to 2^{\mathbb{N}}$ be a map assigning a list of more than $\log_2(n)$ colours to each vertex. Prove that there exists a proper colouring of G such that each vertex uses a colour from its list.

Exercise 4.

Prove that for n large enough, there exists a graph with chromatic number at least $n/2$ and clique number at most $n^{3/4}$.

Exercise 5.

Prove that every n-vertex 3-uniform hypergraph with $m \geq n/3$ edges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3m}}$.

Exercise 6.

Let $p > n > m^2$ be integers with p prime. Let $0 < a_1 < a_2 \dots a_m < p$ be integers. Prove that there exists an integer $0 < x < p$ such that the number

$$(xa_i \mod p) \mod n, \quad \text{for } i \in [m]$$

are pairwise distinct.

Exercise 7. Method of Deferred Decisions

Let G be the random graph $G(2n, 1/2)$. Show that the following inductive procedure produces a perfect matching M of G with probability at least $1/3$: Choose an arbitrary unmatched vertex u, if all neighbours of u are already matched the procedure fails, otherwise match u with an arbitrary unmatched neighbour and recurse on the remaining unmatched vertices.

Exercise 8.

Let H be a graph and let $n > |V(H)|$ be an integer. Suppose that there exists an n-vertex graph L with t edges and containing no copy of H, where $tk > n^2 \ln n$. Prove that there exists a k-colouring of the edges of K_n with no monochromatic copy of H.

★ Exercise 9. Property B in hypergraphs with bounded number of edges

Denote $m(k)$ the minimum number of hyperedges in a k-uniform hypergraph that is not 2-colourable. The goal of this exercise is to prove the following lower bound:

$$\Omega\left(\frac{2^{k-1}k^{1/3}}{\ln(k)^{1/2}}\right) \leq m(k)$$

Let H be a k-uniform hypergraph with $2^{k-1}\ell$ edges, such that there exists $p \in [0, 1/2]$ with $2\ell(1-p)^k < 1/2$ and $2\ell^2 p(1+p)^{k-1} < 1/2$. Show that the following algorithm colours H properly with positive probability:

Step 1 Colour every vertex independently blue or red with probability $1/2$. An edge of H is *dangerous* if it is monochromatic in this first colouring.

Step 2 For every vertex v that belongs to at least one dangerous edge e, change the colour of v with probability p.

1. Let A_e be the event that $e \in E(H)$ monochromatic in the initial and in the final colouring. Prove that $\mathbb{P}[\bigvee_{e \in E(H)} A_e] \leq 2\ell(1-p)^k$.
2. Given two edges $e, f \in E(H)$ such that $e \cap f \neq \emptyset$, let B_{ef} be the event that e is red in the initial colouring and f blue in the final colouring, or e blue in the initial colouring and f red in the final colouring. Prove that $\mathbb{P}[B_{ef}] \leq 2^{2-2k} p(1+p)^{k-1}$. (Hint: partition f into two parts that are initially coloured red and blue)
3. Deduce that the final colouring is proper with positive probability.
4. Conclude.
5. (Optional) How could we improve this bound?

Exercise 10.

1. Prove that for n large enough, every n-vertex directed graph of minimum outdegree $\delta^+ \geq \log_2 n - \frac{\log_2 \log_2 n}{10}$ contains a directed cycle of even length.
2. Using Exercise 9, show that the result still holds when $\delta^+ \geq \log_2 n - \frac{\log_2 \log_2 n}{10}$ for n large enough.