Entropy 2

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Exercise 1.

Let $\mathcal{F}$ be a collection of subsets of $[n]$. Denote p_i the fration of subsets in $\mathcal{F}_i$ that contain i. Prove that

$$|\mathcal{F}| \leq \prod_i p_i^{-pi}(1-p_i)^{-(1-p_i)}$$

Exercise 2.

Let X, Y be random integer-valued variables.

1. Assume that X and Y are independent. Show that $\mathrm{H}(X+Y) \geq \max(\mathrm{H}(X), \mathrm{H}(Y))$
2. If X and Y are no longer independent, can we have $\mathrm{H}(X+Y) < \min(\mathrm{H}(X), \mathrm{H}(Y))$?

Exercise 3.

Let $\mathcal{G}$ be a family of graphs on $2n$ labelled vertices, such that for each $G_i, G_j \in \mathcal{G}$, the graph $G_i \cap G_j$ contains a perfect matching. Show that $|\mathcal{G}| \leq 2^{\binom{2n}{2}-n}$ and that this bound is tight.

Exercise 4.

Let H be a 5-cycle with vertices labelled from 1 to 5. Let G be a graph on n vertices with vertices labelled by $[n]$ and m edges. Recall that an embedding of H in G is a map $\phi : [5] \to [n]$ such that for every $ij \in E(H)$, $\phi(i)\phi(j) \in E(G)$.

Prove that the number of embeddings of H in G is at most $(2m)^{5/2}$.

Exercise 5.

Let A, B, C be finite subsets of $\mathbb{R}$. Prove that

$$|A+B+C|^2 \leq |A+B| \cdot |A+C| \cdot |B+C|$$

where $A+B = \{a+b : a \in A, b \in B\}$.

Exercise 6. Submodularity and Shearer for sums of independent random variable

Let $\mathrm{I}(X;Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X,Y)$ be the mutual information between X and Y.

Data processing Prove that for every deterministic function f, $\mathrm{I}(f(X);Y) \leq \mathrm{I}(X,Y)$

Independence Prove that if Z is independent from (X,Y), $\mathrm{I}((X,Z);Y) = \mathrm{I}(X;Y)$

Submodularity Let X, Y, Z be mutually independent random integers. Prove that $\mathrm{H}(X+Y) + \mathrm{H}(Y+Z) \geq \mathrm{H}(X+Y+Z) + \mathrm{H}(Y)$

Shearer Let X, Y, Z be mutually independent random integers. Prove that $2\,\mathrm{H}(X+Y+Z) \leq \mathrm{H}(X+Y) + \mathrm{H}(X+Z) + \mathrm{H}(Y+Z)$.

Exercise 7.

A Sudoku of size n is an $n^2 \times n^2$ array $(s_{ij})_{i,j\in[n]}$ such that each line, each column and each block $(s_{ij})_{pn<i\leq(p+1)nqn<j\leq(q+1)n}$ for some $p, q \in [n]$ contains every number from $[n^2]$.

1. Using the Poisson paradigm, guess an upper bound on the number of Sudokus of size n.
2. Prove it using the entropy method.