Entropy

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Exercise 1.
Prove the following (in)equalities:

1. Chain rule: $\mathrm{H}(X,Y) = \mathrm{H}(X) + \mathrm{H}(Y|X)$
2. Subadditivity: $\mathrm{H}(X,Y) \leq \mathrm{H}(X) + \mathrm{H}(Y))$
3. Submodularity: $\mathrm{H}(X,Y,Z) + \mathrm{H}(X) \leq \mathrm{H}(X,Y) + \mathrm{H}(X,Z))$

Exercise 2.
Let $m \in \mathbb{N}$ and $S_m = \sum_{k=1}^{10} k^{-m}$. Let X_m be a random variable such that for every $k \in [10]$, $\mathbb{P}(X_m = k) = \frac{1}{S_m k^m}$. Without any calculations, what can you say about the sequence $(\mathrm{H}(X_m))_{m\in\mathbb{N}}$?

Exercise 3.
Let $p \in [0,1]$ and $n \in \mathbb{N}^+$ such that pn is integral. Prove that $\frac{2^{n\,\mathrm{H}(p)}}{8\sqrt{p(1-p)}} \leq \binom{n}{pn}$ using Stirling's formula:

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

1 Counting Latin squares

A Latin square of size n is an $n \times n$ array $(s_{ij})_{i,j\in[n]}$ such that each line and each column contains every number from $[n]$.

Exercise 4. Counting Latin squares using Brégman's result on the permanent
Brégman proved that for every matrix $A \in \{0,1\}^{n\times n}$, the permanent of A is at most $\prod_i (d_i!)^{1/d_i}$, where d_i is the number of ones in the i-th line. Using this result, prove that the number of Latin squares of size n is at most $n!\prod_{k=1}^{n-1}(k!)^{n/k}$.

Hint : Assuming that you already filled the k first lines of the Latin square, count the possible extensions

★ **Exercise 5. Counting Latin squares using entropy**

The goal of this exercise is to prove that the number of Latin squares of size n is at most $\left(\frac{n^2}{e^2+o(1)}\right)^{n^2}$

To do so, consider a random uniform Latin square and reveal its entries in a random order, like we did for matchings. We will use (and admit) that $\int_0^1 \log_2(1+(n-1)z^2 dz = \log_2(n) - 2 + o(1)$.

Hint : To sample the random order, pick uniform independent reals in $[0,1]$