Online edge colouring

Standard edge coloring :

[Vizing '64] $\forall G$, $\Delta \leq \chi'(G) \leq \Delta+1$

[Holyer '81] NP-complete to decide if $\chi'(G) = \Delta$ or $\Delta+1$.

Online problem : different settings

- vertex arrival
- edge arrival

- random order
- adversarial order
- adaptative order

- Δ known vs unknown.

Color the edges progressively without modifications.

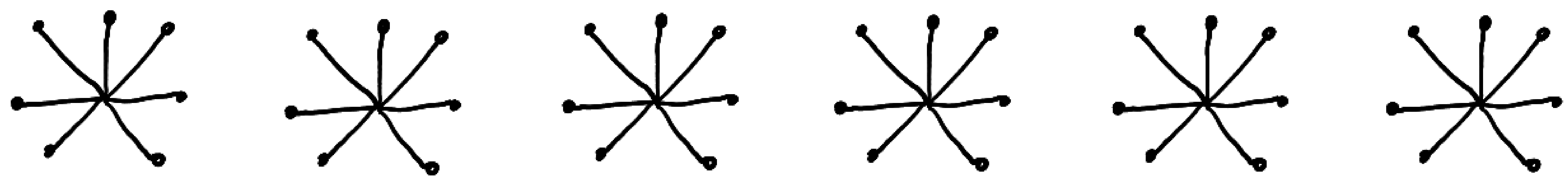
[Bar-Noy, Motwani, Naor '92]:

1. Greedy algorithm: use a color not in $N(e)$

$\leadsto 2\Delta-1$ colors

2. Cannot maintain a $2\Delta-2$ edge coloring when $\Delta = O(\log n)$

Start with β $(\Delta-1)$-stars



For $\beta \geq (\Delta-1)\binom{2\Delta-2}{\Delta-1}$, Δ of them are colored identically, introduce a vertex dominating their centers.

3. Extension to random order of arrival [Bhattacharya, Grandoni, Wajc '21]

$\mathcal{H}$: $\beta = 2\Delta \binom{2\Delta-2}{\Delta-1}\binom{2\Delta-1}{\Delta-1} \leq 4^{O(\Delta)}$ $(\Delta-1)$-stars

$+ v$ adjacent to Δ random stars.

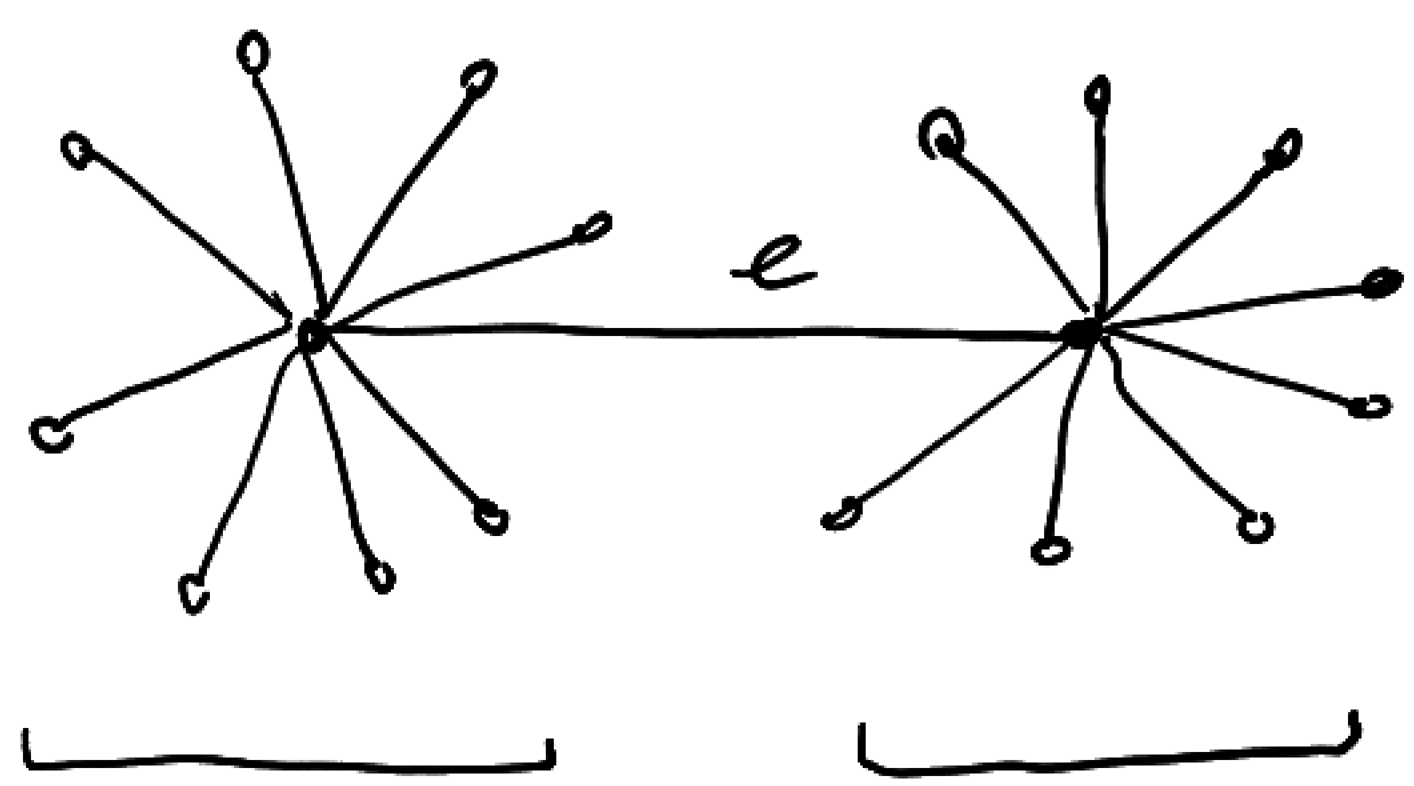
$\mathbb{E}[\#$ of stars here before $v] = 2\Delta\binom{2\Delta-2}{\Delta-1}$ $\leadsto$ pigeon hole.

$\boxed{\text{What competitive ratio can we achieve when } \Delta = \omega(\log n)\text{ ?}}$

- Conjecture : the following algorithm can do $\Delta + O(\sqrt{\Delta} \log n)$ with good probability when $\Delta = \omega(\log n)$

[Color each edge with a random color among the available colors.

Tight :



reveal first , reveal second , reveal e

$k = \Delta + \Omega(\Delta)$ to have constant probability of success.

Random order edge-arrival:

[Aggarwal, Motwani, Shah, Zhu '03]: $(1+o(1))\Delta$ for multigraphs with $\Delta = \omega(n^2)$

[Bahmani, Mehta, Motwani '12]: $1.26\,\Delta$ when $\Delta = \omega(\log n)$

[Bhattacharya, Grandoni, Wajc '21]: $(1+o(1))\,\Delta$ when $\Delta = \omega(\log n)$ using Nibble method.

Standard nibble method for distributed coloring: In each round, each vertex select a random ε fraction of its incident edges. $\forall e$ selected, choose a random color available. If there is a conflict, do nothing.

Why does it work?

- Each edge fails with probability $O(\varepsilon)$

$\rightsquigarrow$ the uncolored subgraph decreases its degree w.h.p at rate $1-\varepsilon$

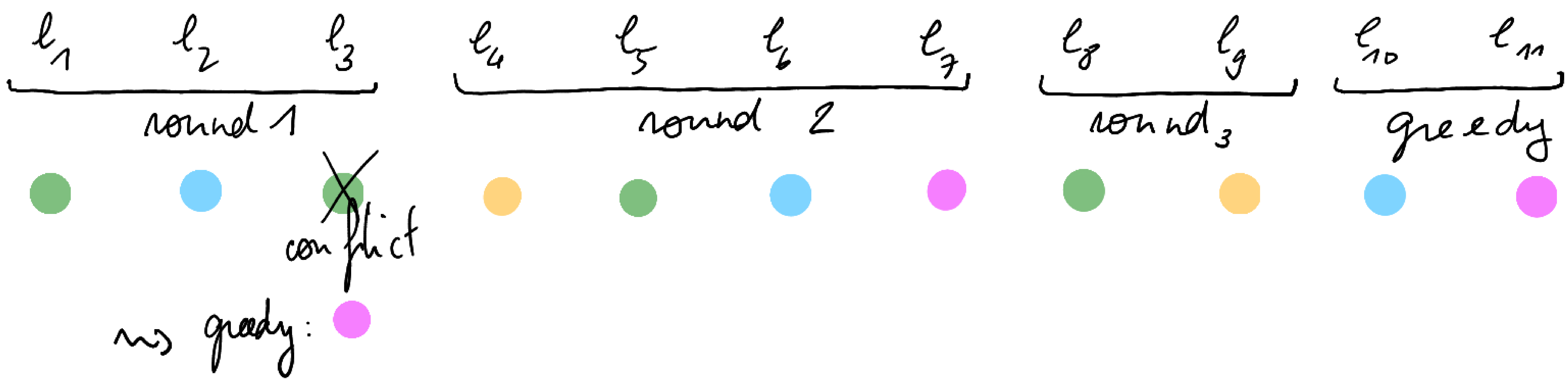
- After $t_\varepsilon = O\left(\frac{\log(1/\varepsilon)}{\varepsilon}\right)$ rounds, uncolored degree $= \text{poly}(\varepsilon)$

we apply a greedy algorithm.

Easier analysis:

- Sample the edges independently at rate ε.
- When conflict, color greedily directly
- If conflict, remove the color of the palette of the vertices.

Online version:



Only difference: no knowledge of the rounds and choices within rounds are not simultaneous.

- Virtual rounds of the appropriate length, update the color palette of the vertices at the end of each round.
- No conflict for the first edge using a color.

Adversarial vertex arrival

[Cohen, Peng, Wajc '19]

- $(1+o(1))\Delta$ for bipartite graphs with one-sided vertex arrival.
- Reduction from online edge coloring to online matching
- When Δ is unknown, cannot do better than $\left(\frac{e}{e-1}\right)\Delta$ even for bipartite graphs, one-sided vertex arrival

[Saberi, Wajc '19]:

- online reduction from general graphs to bipartite graphs against oblivious
- $(1.9+o(1))\Delta$ for general graphs.

Adversarial edge arrival

[Kulkarni, Liu, Sah, Sawhney, Tarnawski 21]

$$\left(\frac{e}{e-1} + o(1)\right)\Delta \quad \text{when} \quad \Delta = \omega(\log n)$$

$\approx 1{,}58$

(Against oblivious)

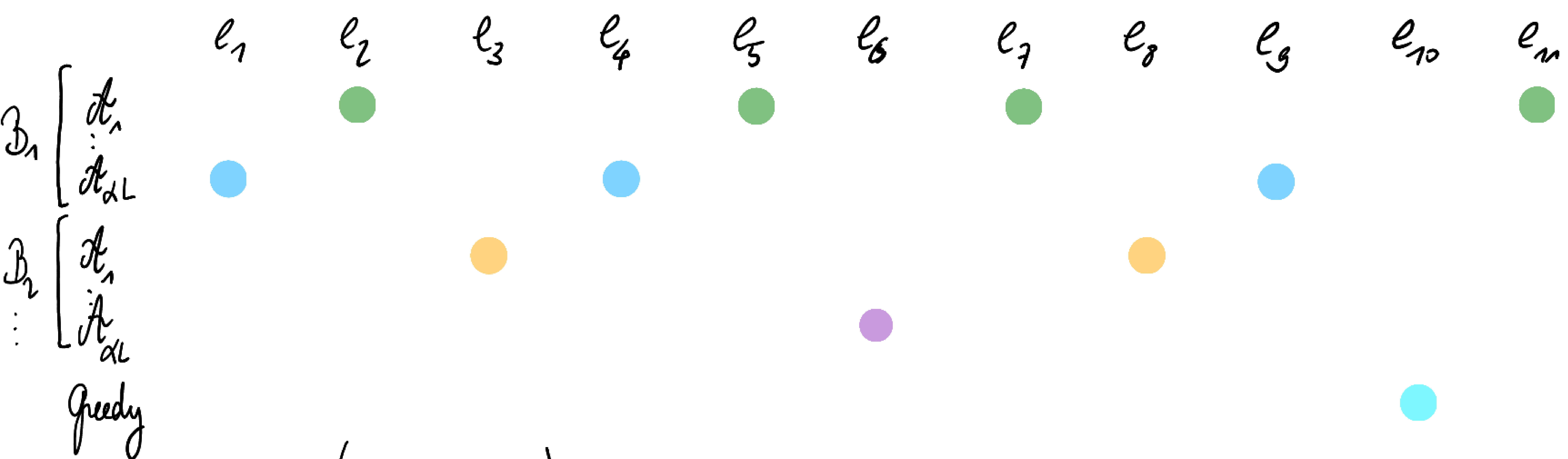
Sketch of the proof:

Reduction to an online matching problem on tree-like graphs

Thm: $\exists$ an online matching algorithm, $\alpha \geq 1$ s.t.
$\forall G$, $\Delta(G) = \Omega(\log n)$, $\mathcal{A}$ matches each edge with marginal probability $\frac{1}{\alpha \Delta(G)}$.

Then, there is an online edge-coloring algorithm $\mathcal{A}'$, s.t.
$\forall G$, $\Delta(G) = \omega(\log n)$, $\mathcal{A}'$ produces a $\left(\alpha + O\left(\left(\frac{\log n}{\Delta}\right)^{1/4}\right)\right)\Delta$ coloring with high probability.

Idea of the proof: inductive run of $\mathcal{A}$.



- $L = O\left(\sqrt{\Delta \log n}\right)$

  H : uncolored subgraph.

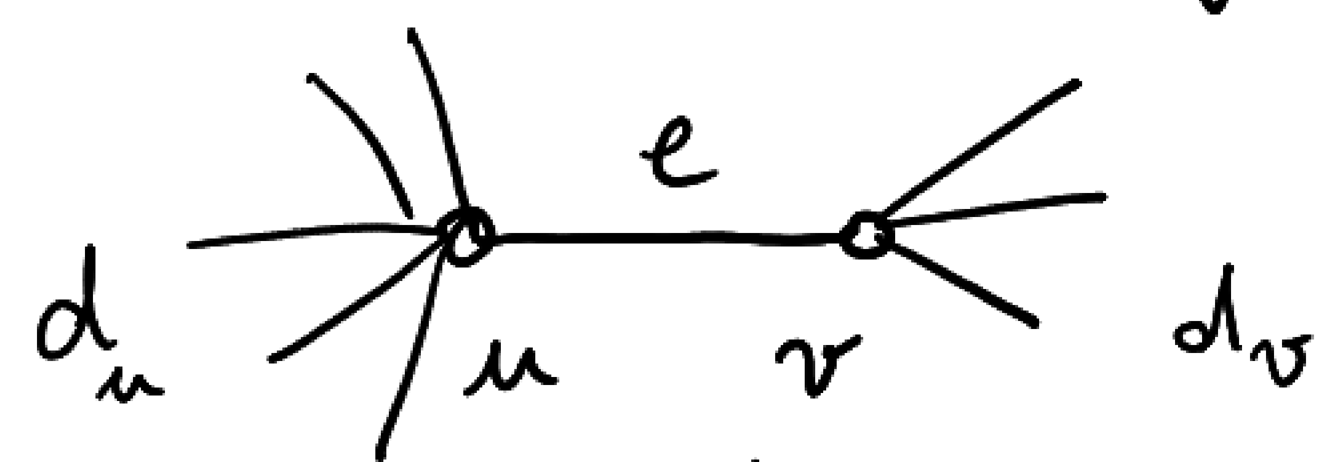
- Claim: If $\Delta(H) = \Delta_i \geq O\left(\sqrt[4]{\Delta^3 \log n}\right)$, After B_i,

$$\mathbb{P}\left[\Delta(H) \geq \Delta_i - L\,(1 - o(1))\right] \leq \frac{1}{n^2}$$

- While $\Delta_i := \Delta - (i-1)L(1-o(1)) \geq O\left(\sqrt[4]{\Delta^3 \log n}\right)$, apply B_i

$\rightsquigarrow$ $\frac{\Delta}{L}$ rounds using αL colors + greedy with $H = o(\Delta)$ $=$ $(\alpha + o(1))\Delta$ colors

Online matching on trees: Match each edge with marginal probability $\frac{1}{c}$



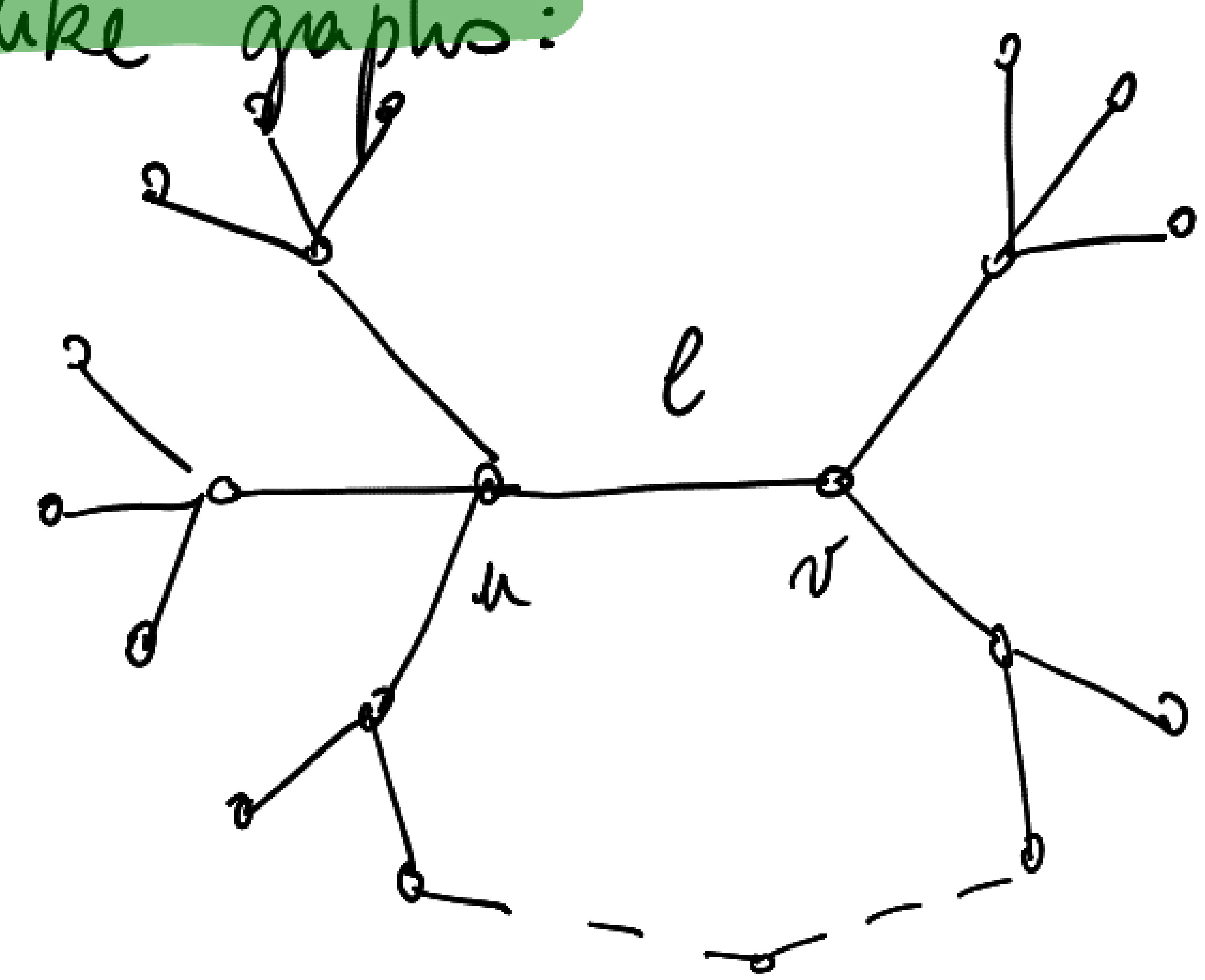
$$\mathbb{P}(u \text{ is not yet matched}) = \left(1 - \frac{d_u}{c}\right)$$

$$\mathbb{P}(v \text{ is not yet matched}) = \left(1 - \frac{d_v}{c}\right)$$

independent in trees

Match uv with probability p s.t. $\left(1-\frac{d_u}{c}\right)\left(1-\frac{d_v}{c}\right)p = \frac{1}{c}$

From trees to treelike graphs:

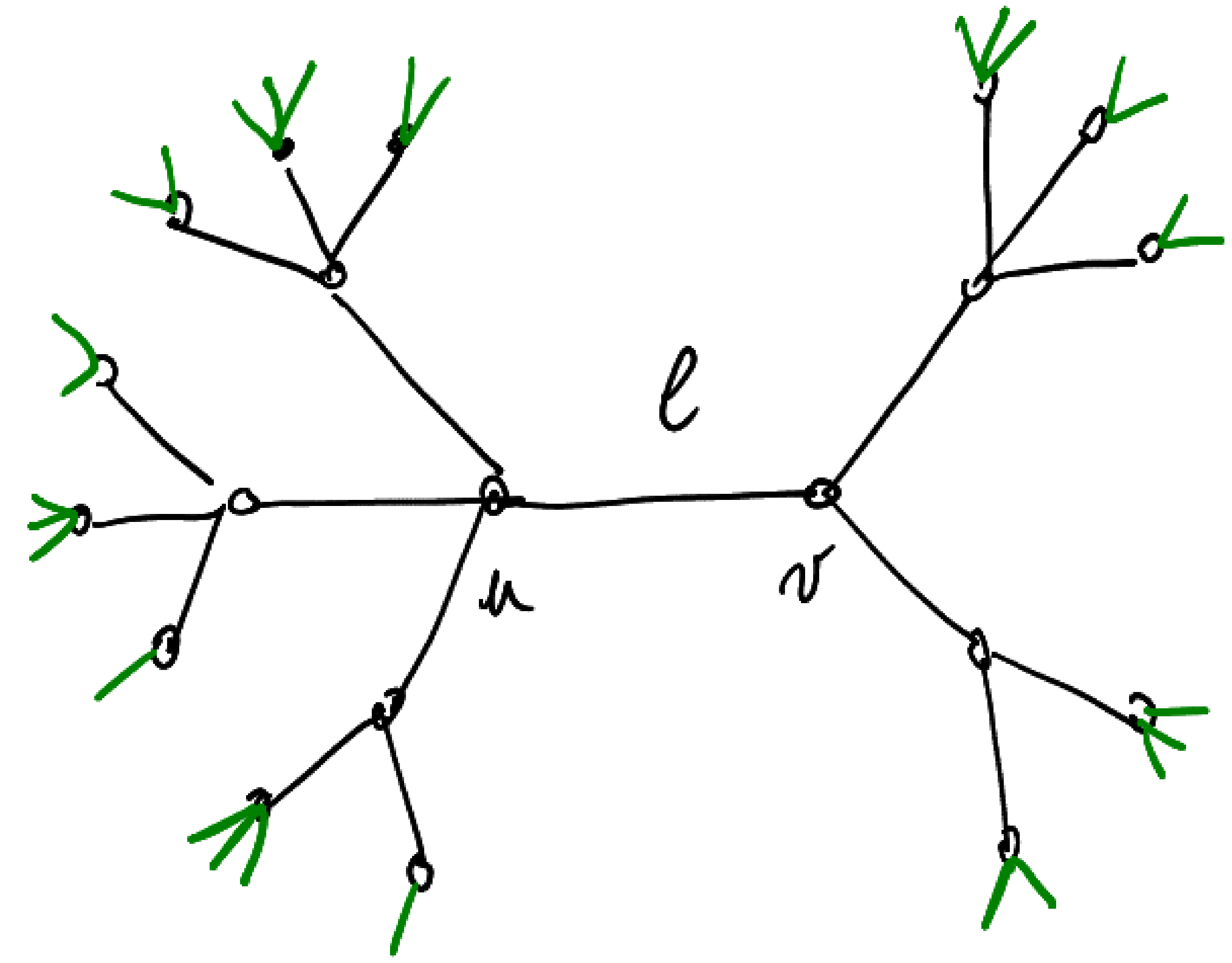


What happens far away should not affect e!

Decay of correlations:

$T = N_d(e)$ for some e.

$\partial T = E(G \setminus T, T)$



Edge matching game: edges of $T \cup \partial T$ appear progressively. If $e_i \in \partial T$, the adversary chooses whether e_i is matched.

Goal of the adversary: minimize $\mathbb{P}(e \text{ is matched})$.

Observation 1: if f is on a $e-g$ path and f appears before g, then g has no impact.

Monotonicity property: for d odd, the optimal strategy of the adversary is to unmatch all edge of ∂T

If $C > \left(\frac{e}{e-1} + o(1)\right)\Delta$, when d grows, $\mathbb{P}(e \text{ is matched} \mid \partial T) \xrightarrow{\text{exp fast}} \frac{1}{C}$

Discussion:

- Reduction to locally tree-like graphs is something more general:
  Given G of max degree Δ and arbitrarily large girth, can one partially edge color G with Δ colors s.t. each edge is colored with probability $1 - o_\Delta(1)$?

  Greedy approach: $\frac{1}{2} - o_\Delta(1)$

  Matching based approach $\frac{e-1}{e} - o_\Delta(1)$

- The $\frac{e}{e-1}$ barrier is tight Kulkarni et. al. matching approach.
  Maybe a "local" algorithm like theirs but with more than two states can work, but in statistical mechanics, these are much harder to anaralyse.
  \+ Monotony seems hard
  ⇝ What is the adversarial strategy on the boundary?

  Connection with Glauber dynamics.

Conclusion and perspectives

1. Bar Noy et. al conjecture true for

   . adversarial vertex arrival . random order edge arrival

   Best known bound for adversarial edge arrival : $\left(\frac{e}{e-1}+o(1)\right)\Delta$

   Conjectured to be true for oblivious adversary but not adaptative.

2. What happens when Δ is unknown?

   [Cohen, Peng, Wajc '19] For adversarial vertex arrival, this is a harder problem, no online algorithm can do better than $\left(\frac{e}{e-1}\right)\Delta$

   Is this something more general?

Thanks !