Kempe recoloring version of Hadwiger's conjecture

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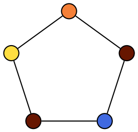
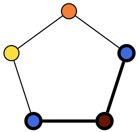
Joint work with Marthe Bonamy, Marc Heinrich and Jonathan Narboni

Recoloring with Kempe changes

Kempe chain (1879)

Maximal bichromatic connected component in G





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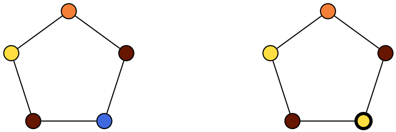

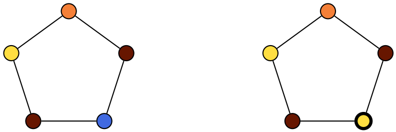

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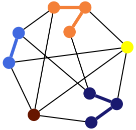
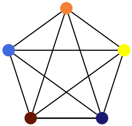
Usual questions

- Are any two k-colorings of a graph G equivalent ?
  Are all k-colorings equivalent to a $\chi(G)$-coloring ?
- How many Kempe changes separate any two k-colorings ?
- Application to sampling : Does the corresponding Markov chain mix well ?

Graph minor

H is a minor of G if H can be obtained be deleting vertices, edges and contracting edges of G





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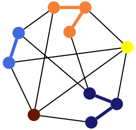

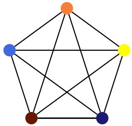

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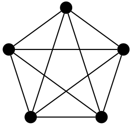
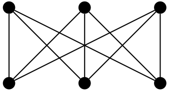
Wagner, Kuratowski 1930

A graph is planar iff K_5-minor and $K_{3,3}$-minor free





Appel, Haken 1976

If G is planar, then $\chi(G) \leq 4$

Robertson, Sanders, Seymour, Thomas 1997

Much simpler proof, but still computer assisted

Hadwiger's conjecture 1943

If G is K_t-minor free then $\chi(G) \leq t - 1$
Proved for $1 \leq t \leq 6$, widely open for $t > 6$

Reconfiguration counterpoint

Meyniel 1978

All 5-colorings of a planar graph are Kempe-equivalent (tight)

Las Vergnas and Meyniel 1981

All 5-colorings of a K_5-minor free graph are Kempe-equivalent

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Conjecture 1 [Las Vergnas and Meyniel 1981]

All the t-colorings of a K_t-minor free graph are Kempe-equivalent

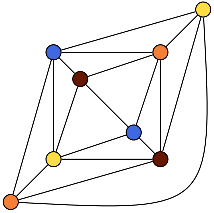
Conjecture 2 [Las Vergnas and Meyniel 1981]

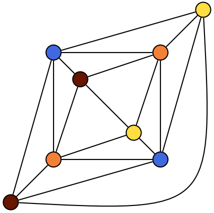
All the t-colorings of a K_t-minor free graph are Kempe-equivalent to a $(t-1)$-coloring

Frozen colorings

Frozen coloring

α is frozen if $\forall i, j$, the graph induced by colors i and j is connected





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Quasi-minor

K_t is quasi-minor of G if there exists $V_1 \sqcup \cdots \sqcup V_t$ such that $\forall i \neq j, G[V_i \cup V_j]$ is connected and $G[V_1, \ldots V_t] = K_t$

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K_t-minor $\Rightarrow$ quasi K_t-minor

Frozen t-coloring $\Rightarrow$ quasi K_t-minor

Last conjecture

Motivation

If G has no K_t minor and all its t-colorings are Kempe equivalent then either

- no frozen t-coloring
- only one t-coloring up to color permutation $\rightsquigarrow$ Hadwiger's conjecture is false

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Conjecture 3 [Las Vergnas and Meyniel 1981]

No K_t-minor $\Rightarrow$ No quasi K_t-minor $\Rightarrow$ No frozen t-coloring

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Conjecture 3 [Las Vergnas and Meyniel 1981]

No K_t-minor $\Rightarrow$ No quasi K_t-minor $\Rightarrow$ No frozen t-coloring

Conjecture 3 holds for

[Las Vergnas and Meyniel '81] $t \leq 5$

[Jørgensen '94] $t = 8$

[Song and Thomas '06] $t = 9$

[Kriesell '21] $t = 10$

To sum up

No K_t-minor implies ...

1. t-recolorable $\wedge \neg$ 3. No quasi-K_t-minor $\longrightarrow$ Hadwiger's conjecture is false

1. $\downarrow$ Assuming Hadwiger

2. Every t-coloring is equivalent to a $(t-1)$-coloring

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Bonamy, Heinrich, L., Narboni '23

- Strongly disproved for large t: $\forall \varepsilon > 0$ and large enough t, $\exists G$ with a frozen t-coloring but no $K_{(\frac{2}{3}+\varepsilon)t}$-minor. This graph admits another t-coloring.

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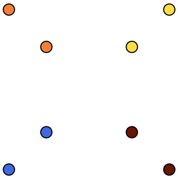
- Strongly disproved for large t: $\forall \varepsilon > 0$ and large enough t, $\exists G$ with a frozen t-coloring but no $K_{(\frac{2}{3}+\varepsilon)t}$-minor. This graph admits another t-coloring.
- Any graph with a quasi-K_t-minor has a $K_{\frac{t}{2}}$-minor

Sketch of proof

Random construction of G_t

- Start with a clique on $V = \{a_1, b_1, \ldots a_t, b_t\}$



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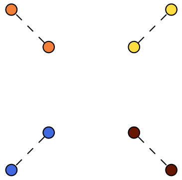

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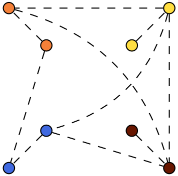

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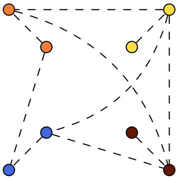
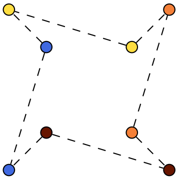


Properties of G_t

- has a frozen t-coloring
- $\mathbb{P}(G_t \text{ has another } t\text{-coloring}) \xrightarrow[t\to\infty]{} 1$
- $\mathbb{P}(G_t \text{ is } K_{(\frac{2}{3}+\varepsilon)t}\text{-minor free}) \xrightarrow[t\to\infty]{} 1$



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Sort the bags in a $K_{(\frac{2}{3}+\varepsilon)t}$-minor

- Bags of size 1 $\rightarrow K_{p_1}$ simple minor
- Bags of size 2 $\rightarrow K_{p_2}$ double minor
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$\mathbb{P}(G_t \text{ is } K_{(\frac{2}{3}+\varepsilon)t}\text{-minor free}) \xrightarrow[t\to\infty]{} 1$ because

- For all $\varepsilon_1 > 0$, $\mathbb{P}(G_t \text{ has no simple } K_{\varepsilon_1 t}\text{-minor}) \xrightarrow[t\to\infty]{} 1$
- For all $\varepsilon_2 > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon_2 t}\text{-minor}) \xrightarrow[t\to\infty]{} 1$
- G_t has no triple $K_{\frac{2}{3}t+1}$-minor

For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no simple } K_{\varepsilon t}\text{-minor}) \xrightarrow[t\to\infty]{} 1$

- Simple K_p-minor = induced K_p
- Given $S \subset V$ of size p,

$$\mathbb{P}(S \text{ induces a } K_p) \leq \left(\frac{3}{4}\right)^{\binom{p}{2}}$$

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- By Union-Bound:

$$\begin{aligned}\mathbb{P}(G_t \text{ has an induced } K_{\varepsilon t}) &\leq \binom{2t}{\varepsilon t}\left(\frac{3}{4}\right)^{\binom{\varepsilon t}{2}} \\ &\leq 2^{2t}\left(\frac{3}{4}\right)^{\binom{\varepsilon t}{2}} \\ &\xrightarrow[t\to\infty]{} 0\end{aligned}$$

For all $\varepsilon > 0$, $\mathbb{P}(G_t \text{ has no double } K_{\varepsilon t}\text{-minor}) \xrightarrow[t\to\infty]{} 1$

A special case of double-minor

- Let S' be a set of pairwise disjoint pairs of vertices, such that $\forall i$, at most one of a_i, b_i is involved in S'.
- $G_t \backslash S'$: contract pairs in S' and remove the rest

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- $\mathbb{P}(G_t \backslash S' \text{ is a clique}) = \left(1 - (\frac{1}{4})^4\right)^{\binom{|S'|}{2}}$

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- For $|S'| = \varepsilon' t$, at most $\binom{2t}{2\varepsilon' t} \cdot (2\varepsilon t)! \leq (2t)^{2\varepsilon t}$ possibilities
- By Union-Bound:

$$\mathbb{P}(\exists \text{ special } S', G_t \backslash S' = K_{\varepsilon' t}) \leq (2t)^{2\varepsilon' t} \left(1 - \frac{1}{4^4}\right)^{\binom{\varepsilon' t}{2}} \xrightarrow[t\to\infty]{} 0$$

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Reducing to the special case

- Let S be a double $K_{\varepsilon t}$-minor
- Greedy special $S' \subset S$: $\forall i$, if a_i and b_i are involved in S, remove the pair containing b_i

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- Greedy special $S' \subset S$: $\forall i$, if a_i and b_i are involved in S, remove the pair containing b_i
- $|S'| \geq \frac{\varepsilon}{3} t$ so take $\varepsilon' = \frac{\varepsilon}{3}$:

$$\mathbb{P}(\exists S \text{ a double } K_{\varepsilon t}\text{-minor}) \leq \mathbb{P}(\exists \text{ a special } S', G_t \backslash S' = K_{\varepsilon' t}) \xrightarrow[t\to\infty]{} 0$$

Open questions

- What it is the infimum c such that for t large enough, there is G with a quasi K_t-minor but no K_{ct}-minor ?

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