The structure of quasi-transitive graphs avoiding a minor

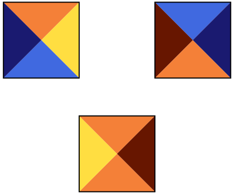
Clément Legrand-Duchesne

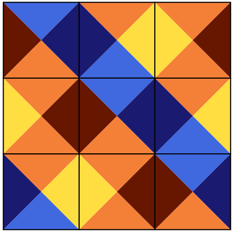
LaBRI, Bordeaux

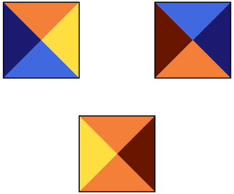
October 13, 2023

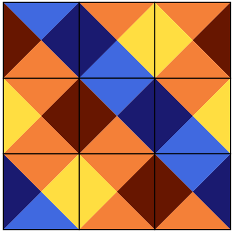
Joint work with Louis Esperet and Ugo Giocanti.

Wang's dominos



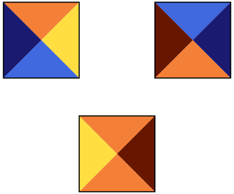


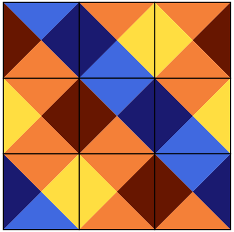




Wang's tiling problem

Entry: A finite family of tiles $\mathcal{T}$
Question: Does there exist a tiling of $\mathbb{Z}^2$ using tiles of $\mathcal{T}$?





Wang's tiling problem

Entry: A finite family of tiles $\mathcal{T}$
Question: Does there exist a tiling of $\mathbb{Z}^2$ using tiles of $\mathcal{T}$?

Theorem

- Wang's tiling problem is undecidable.
- There exist aperiodic sets of tiles

Generalization of Wang's dominos

What about tiling other spaces ?

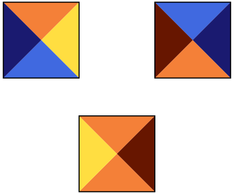
On $\mathbb{Z}$, $\exists$ a tiling $\Leftrightarrow$ $\exists$ a periodic tiling, so the problem is decidable.

What about tiling other spaces ?

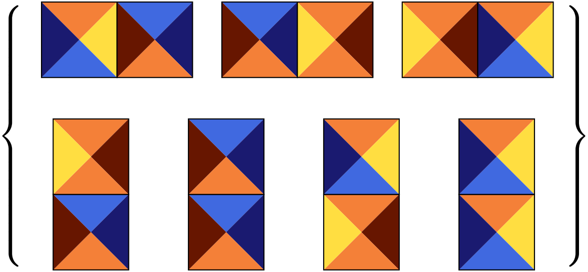
On $\mathbb{Z}$, $\exists$ a tiling $\Leftrightarrow$ $\exists$ a periodic tiling, so the problem is decidable.

Alternative definition of the problem

Entry: k colors and a finite set of forbidden patterns $\mathcal{F}$
Question: Is there a coloring of $\mathbb{Z}^2$ that avoids $\mathcal{F}$?



$\mathcal{F} =$

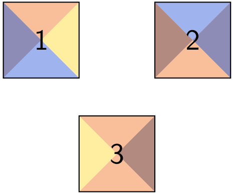


What about tiling other spaces ?

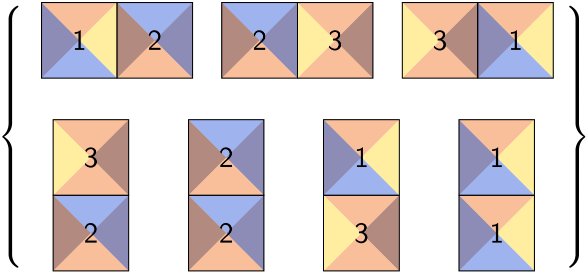
On $\mathbb{Z}$, $\exists$ a tiling $\Leftrightarrow$ $\exists$ a periodic tiling, so the problem is decidable.

Alternative definition of the problem

Entry: k colors and a finite set of forbidden patterns $\mathcal{F}$
Question: Is there a coloring of $\mathbb{Z}^2$ that avoids $\mathcal{F}$?



$\mathcal{F} =$



Goal of this talk

Overview and intuition on different objects

- The domino problem
- Infinite graphs with lots of symmetries
- Tree-decompositions, treewidth and minors
- A small bit of group theory

Natural generalization of $\mathbb{Z}^d$: Cayley graphs

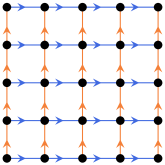
Group presentation of $\Gamma = \langle \Sigma | R \rangle$

- A finite set of generators and their inverses:
  $\Sigma = \{a, a^{-1}, b, b^{-1} \dots\}$
- A set of relations $R = \{aba^{-1}b^{-1}\}$
  finitely presented if R is finite
- The elements are the words on Σ, quotiented by patterns in R

Natural generalization of $\mathbb{Z}^d$: Cayley graphs

Group presentation of $\Gamma = \langle \Sigma | R \rangle$

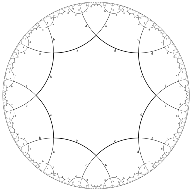
- A finite set of generators and their inverses: $\Sigma = \{a, a^{-1}, b, b^{-1} \ldots\}$
- A set of relations $R = \{aba^{-1}b^{-1}\}$
  finitely presented if R is finite
- The elements are the words on Σ, quotiented by patterns in R



$\langle a, b | aba^{-1}b^{-1} \rangle$

Cayley graphs

Vertices are the elements of the group
Edges are labelled by Σ

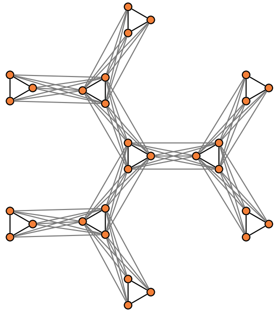


$\langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$

Why Cayley graphs ?

Some examples of Cayley graphs

- $\mathbb{Z}^d$
- The infinite *d*-valent trees and their blow-ups



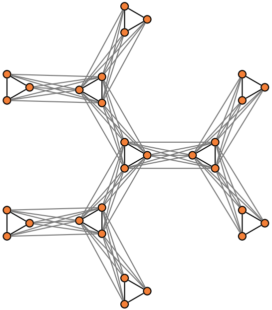
Why Cayley graphs ?

Some examples of Cayley graphs

- $\mathbb{Z}^d$
- The infinite d-valent trees and their blow-ups

Strong structural properties of Cayley graphs

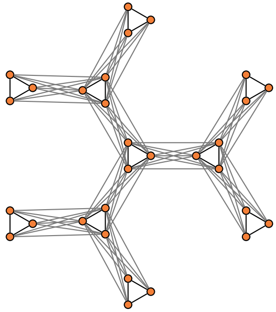
- Regular
- Transitive: For all u, v, $\exists \phi \in Aut(G), u = \phi(v)$
- Strong connections with expanders



Why Cayley graphs ?

Some examples of Cayley graphs

- $\mathbb{Z}^d$
- The infinite d-valent trees and their blow-ups



Strong structural properties of Cayley graphs

- Regular
- Transitive: For all u, v, $\exists \phi \in Aut(G), u = \phi(v)$
- Strong connections with expanders

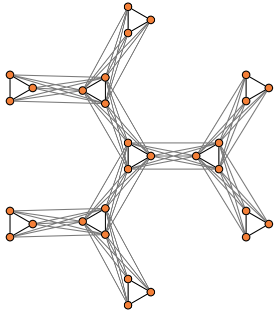
Conjecture [Ballier and Stein 2018]

The domino problem is decidable in a group $\Gamma \Leftrightarrow \Gamma$ is virtually-free

Why Cayley graphs ?

Some examples of Cayley graphs

- $\mathbb{Z}^d$
- The infinite d-valent trees and their blow-ups



Strong structural properties of Cayley graphs

- Regular
- Transitive: For all u, v, $\exists \phi \in Aut(G), u = \phi(v)$
- Strong connections with expanders

Conjecture [Ballier and Stein 2018]

The domino problem is decidable in a group $\Gamma \Leftrightarrow \Gamma$ has a Cayley graph G of bounded treewidth

Definition

A tree decomposition of G is a tree T whose nodes are bags $X_i \subset V(G)$ s. t.

- $\bigcup_i X_i = V(G)$
- $\forall u \in V(G)$ the subgraph of nodes containing u is connected
- $\forall uv \in E(G), \exists X_i, \{u, v\} \subset X_i$

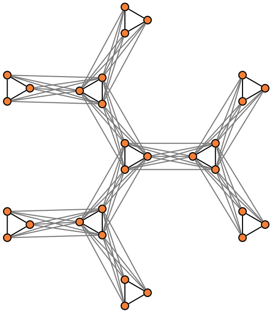
Definition

A tree decomposition of G is a tree T whose nodes are bags $X_i \subset V(G)$ s. t.

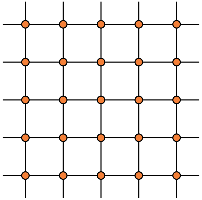
- $\bigcup_i X_i = V(G)$
- $\forall u \in V(G)$ the subgraph of nodes containing u is connected
- $\forall uv \in E(G), \exists X_i, \{u, v\} \subset X_i$

A graph has treewidth at most k if it admits a tree decomposition with bags of size at most $k + 1$

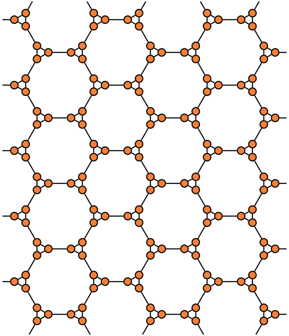
Treewidth of infinite graphs



Bounded treewidth

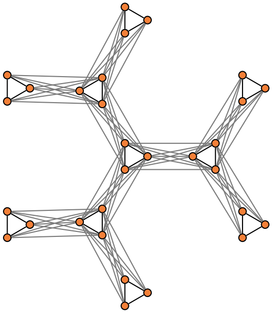


Unbounded treewidth

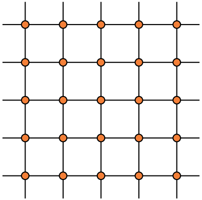


Unbounded treewidth

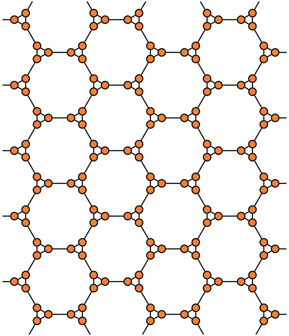
Treewidth of infinite graphs



Bounded treewidth



Unbounded treewidth

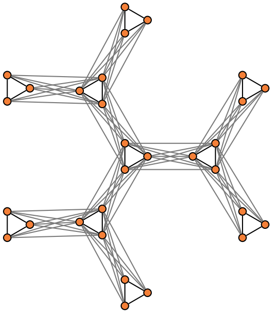


Unbounded treewidth

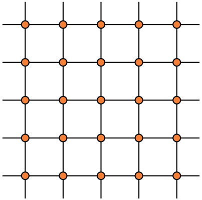
Intuition behind the conjecture

Bounded treewidth $\Rightarrow$ tree-like structure with periodic colorings ✓

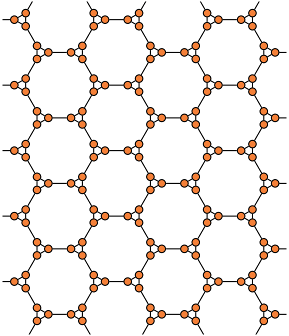
Treewidth of infinite graphs



Bounded treewidth



Unbounded treewidth



Unbounded treewidth

Intuition behind the conjecture

Bounded treewidth $\Rightarrow$ tree-like structure with periodic colorings ✓
Unbounded treewidth $\Rightarrow$ infinite grid-like workspace ?

Vocabulary on tree decomposition

Definitions

Let T be a tree decomposition of G,

Adhesion set: $X_i \cap X_j$ for some $i \neq j$

Adhesion: supremum size of an adhesion set

Vocabulary on tree decomposition

Definitions

Let T be a tree decomposition of G,

Adhesion set: $X_i \cap X_j$ for some $i \neq j$

Adhesion: supremum size of an adhesion set

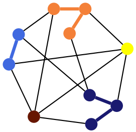
Torso of a bag X_i: graph $G[\![X_i]\!]$ s. t.

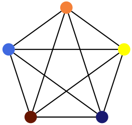
- $G[X_i] \subset G[\![X_i]\!]$
- add all edges uv s. t. u, v in an adhesion of X_i and connected by a path in $G \setminus E[X_i]$.

Crash course on minors

Definition

H minor of G: H can be obtained from G by contracting edges and by deleting vertices and edges.





Proposition

Having bounded treewidth is a minor closed property

Why is G H-minor free ?

Let G be a H-minor-free graph. Then G is piecewise

- too thin to contain H
- almost embeddable on surfaces too simple to contain H as a minor.

Robertson-Seymour's structure theorem on graph minors

Why is G H-minor free ?

Let G be a H-minor-free graph. Then G is piecewise

- too thin to contain H
- almost embeddable on surfaces too simple to contain H as a minor.

Robertson, Seymour 2003

Let H be a fixed graph, $\exists k$, s. t. any H-minor free graph G admits a tree-decomposition with :

- adhesion is at most k,
- torsos are "almost" embeddable in a surface in which H does not embed (too low genus)

Diestel, Thomas 1999

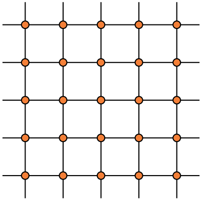
The same holds for **locally-finite** graphs G that exclude some **finite** minor.

What about graphs with many symmetries ?

Definition

G quasi-transitive: $\exists$ a t-coloring of G, s. t. $\forall u, v$ colored identically, $\exists \phi \in Aut(G)$ with $u = \phi(v)$

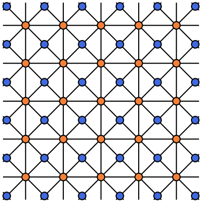
($V(G)$ has finitely many orbits under the action of $Aut(G)$)



What about graphs with many symmetries ?

Definition

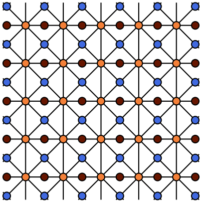
G quasi-transitive: $\exists$ a t-coloring of G, s. t. $\forall u, v$ colored identically, $\exists \phi \in Aut(G)$ with $u = \phi(v)$
($V(G)$ has finitely many orbits under the action of $Aut(G)$)



What about graphs with many symmetries ?

Definition

G quasi-transitive: $\exists$ a t-coloring of G, s. t. $\forall u, v$ colored identically, $\exists \phi \in Aut(G)$ with $u = \phi(v)$
($V(G)$ has finitely many orbits under the action of $Aut(G)$)



Preserve the symmetric structure in T

Definition

Canonical tree decomposition: $\forall \phi \in Aut(G)$, ϕ maps bags on other bags
($Aut(G)$ induces an action on T s. t. $\forall \phi, \forall i$, $\phi(X_i) = X_{i \cdot \phi}$)

Preserve the symmetric structure in T

Definition

Canonical tree decomposition: $\forall \phi \in Aut(G)$, ϕ maps bags on other bags
($Aut(G)$ induces an action on T s. t. $\forall \phi, \forall i$, $\phi(X_i) = X_{i \cdot \phi}$)

Esperet, Giocanti, L. 2023

Let G be a quasi-transitive locally finite graph G avoiding the countable clique as a minor.
Then G admits a **canonical** tree decomposition s. t.

Theorem 1 torsos are finite or planar

Preserve the symmetric structure in T

Definition

Canonical tree decomposition: $\forall \phi \in Aut(G)$, ϕ maps bags on other bags
($Aut(G)$ induces an action on T s. t. $\forall \phi, \forall i$, $\phi(X_i) = X_{i \cdot \phi}$)

Esperet, Giocanti, L. 2023

Let G be a quasi-transitive locally finite graph G avoiding the countable clique as a minor. Then G admits a **canonical** tree decomposition s. t.

Theorem 1 torsos are finite or planar

Theorem 2
- adhesion is at most 3
- torsos are minors of G
- torsos are planar or have bounded treexidth

Hadwiger number

Definition

Hadwiger number of G: supremum of the sizes of its complete minors.

Hadwiger number

Definition

Hadwiger number of G: supremum of the sizes of its complete minors.

Thomassen 1992

Every locally finite quasi-transitive 4-connected graph attains its Hadwiger number.

Definition

Hadwiger number of G: supremum of the sizes of its complete minors.

Thomassen 1992

Every locally finite quasi-transitive 4-connected graph attains its Hadwiger number.

Esperet, Giocanti, L. 2023

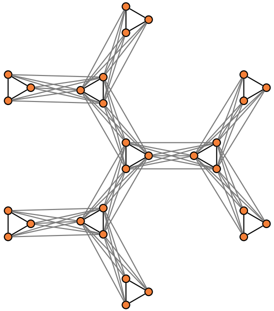
Every locally finite quasi-transitive graph attains its Hadwiger number.

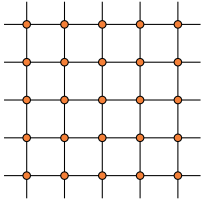
$$\text{“}K_\infty \text{ minor free} \Rightarrow K_t \text{ minor free for some } t\text{”}$$

Graph ends

Definitions

- Ray: infinite one-way path in G
- Two rays r_1 and r_2 are equivalent if $\forall$ finite subgraph C of G, $\exists$ a connected component of $G \setminus C$ intersecting r_1 an infinite number of time, and r_2 too
- End of G: equivalence class of rays

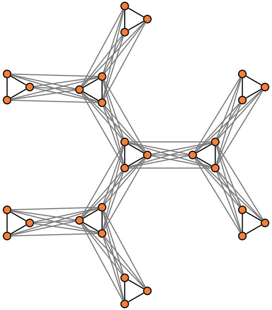


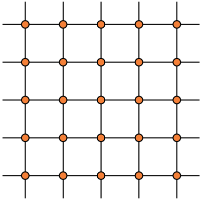


Graph ends

Definitions

- Ray: infinite one-way path in G
- Two rays r_1 and r_2 are equivalent if $\forall$ finite subgraph C of G, $\exists$ a connected component of $G \setminus C$ intersecting r_1 an infinite number of time, and r_2 too
- End of G: equivalence class of rays
- Thickness of an end: supremum in $\mathbb{N} \cup \{\infty\}$ of the number of pairwise disjoint rays living in it

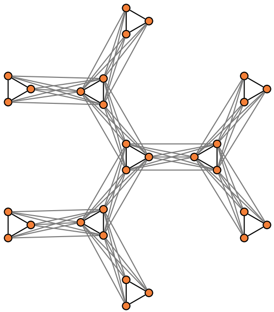


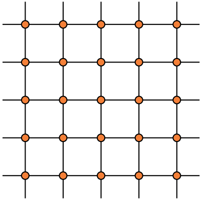


Graph ends

Definitions

- Ray: infinite one-way path in G
- Two rays r_1 and r_2 are equivalent if $\forall$ finite subgraph C of G, $\exists$ a connected component of $G \setminus C$ intersecting r_1 an infinite number of time, and r_2 too
- End of G: equivalence class of rays
- Thickness of an end: supremum in $\mathbb{N} \cup \{\infty\}$ of the number of pairwise disjoint rays living in it



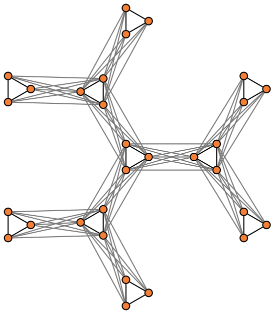


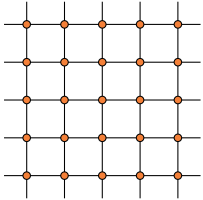
Hopf 1944 & Diestel, Jung, Möller 1993

A quasi-transitive graph has 0,1,2 or an infinite number of ends

Definitions

- A finite set C separates two ends if they have an infinite number of vertices in distinct components of $G \setminus C$
- A graph G is vertex-accessible if there is a $k < \infty$ s. t. any two ends can be separated by a set of size k.

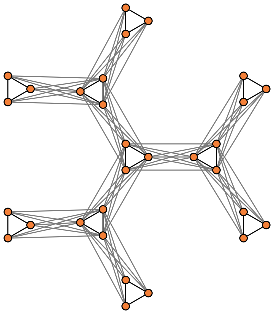




Separating the ends

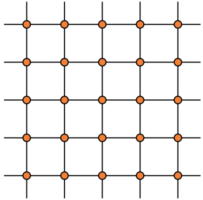
Definitions

- A finite set C separates two ends if they have an infinite number of vertices in distinct components of $G \setminus C$
- A graph G is vertex-accessible if there is a $k < \infty$ s. t. any two ends can be separated by a set of size k.



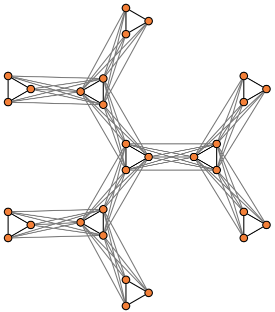
Dunwoody 2007

Planar quasi-transitive graphs are vertex-accessible.



Definitions

- A finite set C separates two ends if they have an infinite number of vertices in distinct components of $G \setminus C$
- A graph G is vertex-accessible if there is a $k < \infty$ s. t. any two ends can be separated by a set of size k.

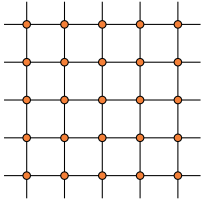


Dunwoody 2007

Planar quasi-transitive graphs are vertex-accessible.

Esperet, Giocanti, L. 2023

Quasi-transitive graphs K_∞-minor free graphs are vertex-accessible.



Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end $\Leftrightarrow$ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings' inductive decomposition terminates

Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end $\Leftrightarrow$ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings' inductive decomposition terminates

Thomassen, Woess 1993 A group is accessible $\Leftrightarrow$ one of its Cayley graphs is vertex-accessible

Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end $\Leftrightarrow$ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings' inductive decomposition terminates

Thomassen, Woess 1993 A group is accessible $\Leftrightarrow$ one of its Cayley graphs is vertex-accessible

$\Gamma = \langle \Sigma | R \rangle$ finitely presented: R finite
Droms 2006 Finitely generated planar groups are finitely presented

Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end $\Leftrightarrow$ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings' inductive decomposition terminates

Thomassen, Woess 1993 A group is accessible $\Leftrightarrow$ one of its Cayley graphs is vertex-accessible

$\Gamma = \langle \Sigma | R \rangle$ finitely presented: R finite
Droms 2006 Finitely generated planar groups are finitely presented
Dunwoody 1985 Finitely presented groups are accessible

Decomposing and presenting groups

Stallings 1972

Γ a finitely generated group. Γ has more than one end ⇔ Γ can be decomposed as a product of two groups (amalgamated free-product or HNN-extension over a finite group)

Definition

Γ accessible: Stallings' inductive decomposition terminates

Thomassen, Woess 1993 A group is accessible ⇔ one of its Cayley graphs is vertex-accessible

$\Gamma = \langle \Sigma | R \rangle$ finitely presented: R finite
Droms 2006 Finitely generated planar groups are finitely presented
Dunwoody 1985 Finitely presented groups are accessible

Esperet, Giocanti, L. 2023 Finitely generated K_∞ minor free groups are accessible and finitely presented

Domino conjecture on groups avoiding a minor

Aubrun, Barbieri, Moutot 2019

For any $g \geq 1$, the fundamental group of the closed orientable surface of genus g has undecidable domino problem

Domino conjecture on groups avoiding a minor

Aubrun, Barbieri, Moutot 2019

For any $g \geq 1$, the fundamental group of the closed orientable surface of genus g has undecidable domino problem

Bungaard, Nielsen 46 & Fox 52

One-ended planar groups contain the fundamental group of a closed orientable surface as a subgroup of finite index

Domino conjecture on groups avoiding a minor

Aubrun, Barbieri, Moutot 2019

For any $g \geq 1$, the fundamental group of the closed orientable surface of genus g has undecidable domino problem

Bungaard, Nielsen 46 & Fox 52

One-ended planar groups contain the fundamental group of a closed orientable surface as a subgroup of finite index

Esperet, Giocanti, L. 2023

The domino conjecture holds in groups with no K_∞-minor

Key ideas to take away

- Among quasi-transitive graphs, planar graphs and graphs excluding a minor are much alike
- For a quasi-transitive graph, K_∞-minor free $\Rightarrow$ K_t minor free for some t

Key ideas to take away

- Among quasi-transitive graphs, planar graphs and graphs excluding a minor are much alike
- For a quasi-transitive graph, K_∞-minor free $\Rightarrow$ K_t minor free for some t

Thanks!